Linear Algebra, Part 1

Lecture 2-2: Analytic Geometry

1 Pythagoras's theorem

Pythagoras's theorem was discovered in ancient times. The theorem relates the lengths of the sides of a right triangle. Consider Figure 1. The theorem states that $c^2 = a^2 + b^2$; in words, the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the other two sides.

It's important to note that Pythagoras's theorem is an "if-and-only-if" statement. That is, if the triangle is a right triangle, then the lengths of the sides satisfy the relation $c^2 = a^2 + b^2$. Conversely, if the lengths of the sides of a triangle satisfy the relation $c^2 = a^2 + b^2$, then the triangle contains a right angle, which is opposite the longest side of the triangle. We shall prove both of these statements later in this section.

In brief, if a triangle is a right triangle, then the lengths of its sides satisfy $c^2 = a^2 + b^2$. If the sides of a triangle satisfy $c^2 = a^2 + b^2$, then the triangle is a right triangle. These two statements can be combined into one statement as follows: A triangle is a right triangle if and only if the lengths of its sides are related by Pythagoras's theorem: $c^2 = a^2 + b^2$.

Therefore, when you read a mathematical statement that contains "if and only if," you will understand that it stands for two separate statements. In terms of logical structure, the first statement is of the form "A implies B," where A and B stand for separate clauses of the sentence, and the second statement is of the form "B implies A." The combined statement is of the form "A is equivalent to B," or, equivalently, "A and B imply each other."

Symbolically, the first statement is of the form "$A \implies B$", the second statement is of the form "$B \implies A$," and the combined statement is of the form "$A \iff B$."

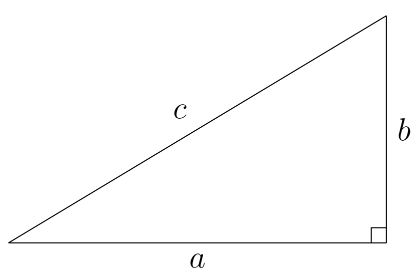

Figure 1: A triangle is a right triangle if and only if the lengths of its sides are related by Pythagoras's theorem: $c^2 = a^2 + b^2$.

EXAMPLE 1

Using Pythagoras's theorem to determine unknown sides of right triangles

(a) If the two shorter sides of a right triangle have lengths 5.7 cm and 3.2 cm, determine the length of the hypotenuse.

(b) If the hypotenuse of a right triangle has a length of 7.1 m and another side of the triangle has length 4.3 m, determine the length of the third side.

SOLUTION

(a) Using the notation of Figure 1, let $a = 5.7$ and $b = 3.2$. Then the length of the hypotenuse c is given by Pythagoras's theorem:

$$\begin{aligned}
c^2 &= a^2 + b^2 \\
c^2 &= 5.7^2 + 3.2^2 \\
c^2 &= 32.49 + 10.24 \\
c^2 &= 42.73 \\
c &= \sqrt{42.73} \\
c &= 6.5368\ldots
\end{aligned}$$

In most scientific applications, it does not make sense to quote more significant digits than are present in the given number that has the least number of significant digits. Since two significant digits are present in each of the given numbers, it is reasonable to quote the final result as

$$c \approx 6.5$$

The units of c are the same as the units of a and b, which were not specified.

(b) Using the notation of Figure 1, let $c = 7.1$ and $b = 4.3$. Then the length of the third side a is given by Pythagoras's theorem:

$$\begin{aligned}
c^2 &= a^2 + b^2 \\
a^2 &= c^2 - b^2 \\
a^2 &= 7.1^2 - 4.3^2 \\
a^2 &= 50.41 - 18.49 \\
a^2 &= 31.92 \\
a &= \sqrt{31.92} \\
a &= 5.6497\ldots \\
a &\approx 5.6
\end{aligned}$$

EXAMPLE 2

Determining whether a triangle is a right triangle

Determine whether each triangle is a right triangle.

(a) a triangle with side lengths of 3 cm, 5 cm, and 7 cm

(b) a triangle with side lengths of 5 m, 12 m, and 13 m

SOLUTION

(a) Label the side lengths a, b, and c in the conventional way, so that c represents the longest side. Then calculate both c^2 and $a^2 + b^2$ to see if they are equal.

$$a = 3, \qquad b = 5, \qquad \text{and} \qquad c = 7$$

$$\begin{aligned} c^2 &= 7^2 \\ c^2 &= 49 \end{aligned}$$

$$\begin{aligned} a^2 + b^2 &= 3^2 + 5^2 \\ a^2 + b^2 &= 9 + 25 \\ a^2 + b^2 &= 34 \end{aligned}$$

A triangle is a right triangle if and only if its side lengths satisfy Pythagoras's relation, $a^2 + b^2 = c^2$. Because the side lengths in this triangle do not satisfy this relation, this triangle is not a right triangle.

(b) Follow the same strategy as in Part (a).

$$a = 5, \qquad b = 12, \qquad \text{and} \qquad c = 13$$

$$\begin{aligned} c^2 &= 13^2 \\ c^2 &= 169 \end{aligned}$$

$$\begin{aligned} a^2 + b^2 &= 5^2 + 12^2 \\ a^2 + b^2 &= 25 + 144 \\ a^2 + b^2 &= 169 \end{aligned}$$

Pythagoras's relation is satisfied, so this triangle is a right triangle.

There is a generalization of Pythagoras's theorem, called the cosine law, which we could use to determine the angles in each of the triangles in Example 1. We shall leave this for another time.

PLAY!

Pythagorean triples

A Diophantine equation is an equation for which the variables are only allowed to be integers, and for which the number of variables is more than one. If we consider the three symbols in Pythagoras's theorem to be variables that are only allowed to be integers, then it can be considered a Diophantine equations. The solutions to this special form of Pythagoras's theorem are called Pythagorean triples, and they have interesting properties, and form interesting patterns when plotted.

The most famous example of a Pythagorean triple is $(3, 4, 5)$, which was known to ancient mathematicians in many lands, long before Pythagoras's life. (It is typical for Pythagorean triples to be quoted in this form, where the first two numbers represent the values of a and b, and the third number represents the value of c.) Based on our knowledge of similar triangles, we can conclude that multiplying the length of each side of this triangle by any whole number will produce a triangle that is still right, and therefore multiplying each number in a Pythagorean triple by a whole number results in another Pythagorean triple. Check it for yourself: Are $(6, 8, 10)$, $(9, 12, 15)$, and so on, all Pythagorean triples? Can you prove this using algebra?

Thus, every time you discover a Pythagorean triple that is "primitive" (for which there are no common factors in the numbers), you can immediately determine an infinite number of related ones. This focusses attention on primitive Pythagorean triples.

You might like to try discovering some primitive Pythagorean triples on your own. Once you have determined a number of them, you might like to explore some of the many interesting properties of Pythagorean triples.

If you find explorations such as this interesting, you might like to carry out further interesting studies in the branch of mathematics known as *number theory*.

Now that you've worked through a number of examples, let's prove Pythagoras's theorem. There are many proofs of this very famous theorem, but I'll present you with two that I particularly like. The first is based on Figures 2 and 3, which are adapted from Figure 12 on Page 62 of *An Introduction to the History of Mathematics*, Fourth Edition, by Howard Eves (Holt, Rinehart, and Winston, 1976).

The diagram in Figure 2 is constructed as follows. First the outer square is constructed. Then the four slanted lines are constructed to produce the inner quadrilateral, whose sides all have the same length, labelled c. I assert that the inner quadrilateral is also a square, although that requires some proof. Convince yourself that the inner quadrilateral is indeed a square; I'll provide a proof later, after the main argument is complete.

Now one way to proceed is to consider the diagram in Figure 2 to consist of four equivalent right triangles within the large outer square. The four triangles can be rearranged within the large outer square as in Figure 3. (For example, you can imagine a square sheet of paper with side lengths $a + b$, and then you can imagine cutting the four right triangles out of the large square with scissors, then rearranging the four right triangles as in Figure 3.) If we then ignore the triangles in each of the two diagrams, the area of the remaining regions are the same. Thus, the area of the square with side lengths c in Figure 7 is equal to the sum of

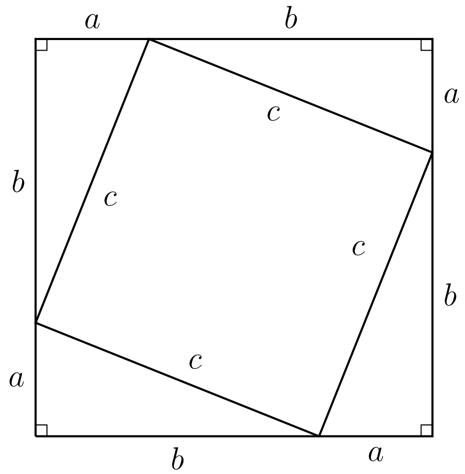

Figure 2: This figure is used in the text to provide a simple proof of Pythagoras's theorem.

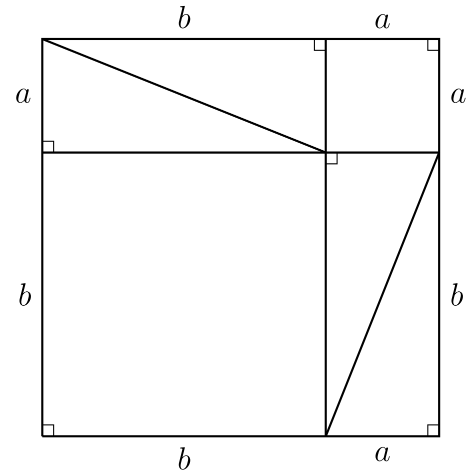

Figure 3: If you think of the diagram on the left as four congruent right triangles within a large square, then the diagram on the right shows the four triangles rearranged within the same square. Comparing the areas of the two diagrams results in a proof of Pythagoras's theorem.

the areas of the two squares with side lengths a and b respectively in Figure 3. Thus,

$$c^2 = a^2 + b^2$$

An alternative calculation is to use just the diagram in Figure 2, write two expressions for the total area of the outer square, and then equate the two expressions. The first expression for the area A of the outer square is just the length times the width:

$$A = (a + b)(a + b)$$

The second expression for the area A of the outer square is the sum of the area of the inner square and the four right triangles:

$$A = (c)(c) + 4\left(\frac{1}{2}(a)(b)\right)$$

Equating the two expressions for the area A of the outer square and simplifying results in

$$\begin{aligned}(a + b)(a + b) &= (c)(c) + 4\left(\frac{1}{2}(a)(b)\right)\\ a^2 + 2ab + b^2 &= c^2 + 2ab\\ a^2 + b^2 &= c^2\end{aligned}$$

To complete the proof, we must go back and convince ourselves that the inner quadrilateral with side lengths c in Figure 2 is a square. We can do this by arguing about the angles in the diagram; see Figure 4, which is a copy of Figure 2, but with some of the angles labelled. First note that the four right triangles are congruent, by construction. Within each right triangle, the sum of the three angles is 180°, so

$$\begin{aligned} A + B + 90^\circ &= 180^\circ \\ A + B &= 90^\circ \end{aligned}$$

Now consider the vertex nearest $\angle C$ in the upper part of the diagram. The three angles at this vertex, A, B, and C fill one side of the horizontal line through this vertex, and so

$$A + B + C = 180^\circ$$

Subtracting the equation $A + B = 90^\circ$ from the one on the previous line, we obtain

$$C = 90^\circ$$

The same argument can be repeated at each of the vertices of the inner quadrilateral, and this proves that all four angles at the vertices of the inner quadrilateral are right angles. Thus, the inner quadrilateral is a square.

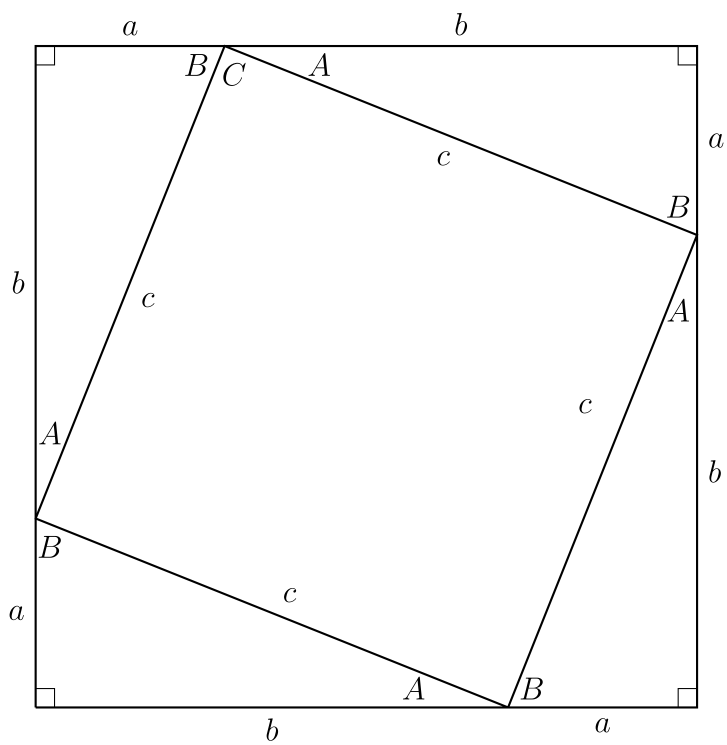

Figure 4: This figure is used in the text to help complete the proof of Pythagoras's theorem. The angles A, B, and C are labelled to facilitate a proof that the inner quadrilateral, with sides labelled c, is a square.

To summarize, what we have proved so far is that if a triangle is a right triangle, then the lengths of the sides of the right triangle are related by $c^2 = a^2 + b^2$. We must now prove the converse statement: If the Pythagoras relationship is satisfied, then the triangle is right.

To play devil's advocate for a moment, perhaps there are other triangles besides right triangles for which the same relationship is satisfied? Let's think about this for a moment.

Imagine that you make a model of a triangle out of sticks. Take three sticks whose lengths are just right so that they can form a right triangle. Now can you imagine somehow disassembling the triangle and putting the sticks together again to form a triangle that is not a right triangle? This doesn't seem possible, and a little bit of play might convince you that it is indeed not possible.

But maybe there is a way to select sticks whose lengths satisfy Pythagoras's relation $c^2 = a^2 + b^2$, and yet the sticks can be put together to form a triangle that is not a right triangle. The discussion of the previous paragraph makes me doubtful, but being doubtful is not a proof. Nevertheless, the idea of the previous paragraph can be used to construct a formal proof, as follows.

Figure 5: The diagrams in this figure are used in the text to facilitate a proof of the converse part of Pythagoras's theorem. The diagram on the left shows a triangle with sides a, b, and c that satisfy the relation $a^2 + b^2 = c^2$, but it is not assumed in advance that this triangle is a right triangle. The diagram on the right shows a right triangle that has been constructed to have sides adjacent to the right angle equal to a and b. By Pythagoras's theorem, the hypotenuse of ΔDEF has length $c = \sqrt{a^2 + b^2}$. Using the SSS theorem of triangle geometry, it follows that ΔABC and ΔDEF are congruent, and the converse part of Pythagoras's theorem follows from this, as shown in the text.

Consider Figure 5. The diagram on the left shows ΔABC whose sides have lengths a, b, and c that satisfy the relation $a^2 + b^2 = c^2$. Note that we are not assuming in advance that ΔABC is a right triangle. Now construct a second triangle, ΔDEF, which is shown on the right side of Figure 5, which has a right angle and for which the sides adjacent to the right angle are the same values a and b that are found in ΔABC. Because ΔDEF is a right triangle, we can use Pythagoras's theorem to determine the length of its hypotenuse, which we shall temporarily denote by d:

$$d^2 = a^2 + b^2$$

Comparing the previous equation with the relation known to be satisfied in ΔABC (that is, $a^2 + b^2 = c^2$), we can conclude that

$$d^2 = c^2$$

and therefore

$$d = c$$

Thus, the lengths of the three sides of ΔABC are equal to the lengths of the corresponding sides of ΔDEF, and so the two triangles are congruent, by the SSS theorem of triangle geometry. This means that corresponding angles of the two triangles are also equal, which means that $\angle BCA = 90°$, and therefore ΔABC is a right triangle. Thus, we conclude that if the lengths of the sides of a triangle satisfy the relation $c^2 = a^2 + b^2$, then the triangle is a right triangle, with the angle opposite the side with length c being the right angle.

KEY CONCEPT

Pythagoras's theorem

If a triangle with side lengths a, b, and c is a right triangle, with the right angle opposite the side with length c, then

$$a^2 + b^2 = c^2$$

Conversely, if a triangle has sides with lengths a, b, and c that satisfy the relation $c^2 = a^2 + b^2$, then the triangle is a right triangle, with the angle opposite the side with length c being the right angle.

Some people think that mathematics is abstract and divorced from reality, and some of the very abstract branches of mathematics appear to be so at first glance. However, history shows that branches of mathematics that initially seem to have no practical use because of their extreme abstraction sometimes eventually become extremely useful to practicing scientists, even centuries after their discovery.

Mathematics may be abstract, but much of it is born from a deep consideration of our experiences with the world. Geometry is particularly amenable to connection with the world, but because such connections are not often exposed in formal mathematics courses, it may be surprising to learn that Pythagoras's theorem is intimately connected with mechanics, as discussed in the "Making Connection" box.

MAKING CONNECTIONS

A mechanical perspective on Pythagoras's theorem (for those with the requisite physics knowledge)

The proof of Pythagoras's theorem presented in this section was guided by geometric reasoning, but there is an interesting mechanical perspective on Pythagoras's theorem. Imagine a hollow, wedge-shaped container, with a cross-section in the shape of a right triangle, as illustrated in Figure 6. The container is filled with fluid (such as air or water), is free to rotate about the attached vertical stick (indicated by the red arrow), and is surrounded by a still fluid.

It is an experimental fact that the container does not rotate under the equilibrium conditions described in the previous paragraph. Yet the fluid inside the container exerts forces on each of the faces of the container, thanks to fluid molecules colliding with them. The fact that the container does not rotate means that the net torque about the red axis due to these forces is zero. Let us calculate this net torque.

We can assume that the net force on each face of the container due to the many molecular collisions can be modelled by a single arrow centred on each face. The forces on the upper and lower faces do not exert torques about the red axis and can be ignored. The force on each of the three vertical faces is proportional to the area of the face; thus, $F_a = kah$, $F_b = kbh$, and $F_c = kch$.

The torque of each force about the red axis is the product of the force and the minimum distance between the line containing the force arrow rotation axis (which can be measured perpendicular to the line through the force arrow). Torques that tend to turn the container counter-clockwise are considered to be positive by convention, and torques that tend to turn the container clockwise are considered to be negative.

The net torque on the container is therefore

$$-kah \cdot \frac{a}{2} - kbh \cdot \frac{b}{2} + kah \cdot \frac{c}{2} = 0$$

Simplifying the previous equation by cancelling the common factors $kh/2$, we obtain

$$\begin{aligned} -a^2 - b^2 + c^2 &= 0 \\ c^2 &= a^2 + b^2 \end{aligned}$$

And there we have it, a delightful connection between mechanics and geometry. You can find more detailed discussions of this mechanical perspective of Pythagoras's theorem in the books *The Application of Mechanics to Geometry*, by Boris Kogan (University of Chicago Press, 1974), and *The Mathematical Mechanic*, by Mark Levi (Princeton University Press, 2009).

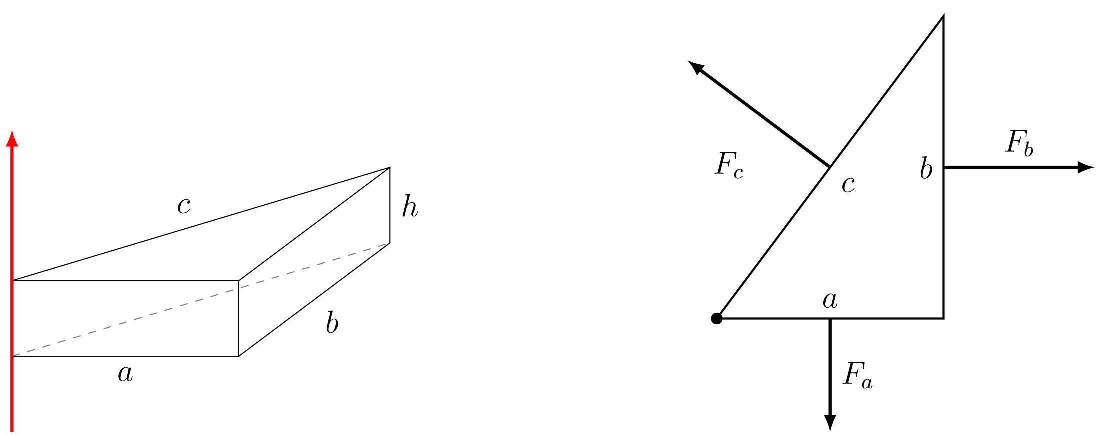

Figure 6: The diagram on the left shows a hollow wedge-shaped container that is filled with fluid. The wedge is free to rotate about a vertical axis that is indicated by the blue arrow. The diagram on the right is an overhead view of the wedge, showing that the cross-section of the container is a right triangle. The net forces F_a, F_b, and F_c exerted by the fluid on each of the vertical faces is indicated by the arrows, and are not to scale.

DIGGING DEEPER

A special case of the triangle inequality

A commonly-heard phrase is that "the shortest distance between two points is a straight line." This can be stated more precisely in mathematical language in a number of ways; a simple way is that the sum of the lengths of any two sides of a triangle is greater than the length of the third side. (Moving along one side of a triangle is the "straight-line" path, whereas moving along the other two sides of the same triangle represents a longer path.) The technical term for this fact is called *the triangle inequality*, and it can be expressed in various forms at various levels of sophistication.

In the context of right triangles, we can express the triangle inequality as

$$c \leq a + b$$

which is equivalent to (making use of Pythagoras's theorem)

$$\sqrt{a^2 + b^2} \leq a + b$$

where a and b are non-negative real numbers. If a and b are allowed to be negative (so that we are no longer in the context of triangle geometry, but in the context of real numbers), then we can modify the statement of this version of the triangle inequality as

$$\sqrt{a^2 + b^2} \leq |a| + |b|$$

The previous inequality is a purely numerical statement, yet we obtained it by geometrical considerations, which is interesting, and illustrates the utility of geometry. The earlier statement, in the context of triangle geometry, is "clearly correct," but is this latest version, where we are allowing arbitrary real numbers, also correct? It's worthwhile to play with this inequality to get a feel for it; substitute a variety of positive and negative numbers and zero for a and b and check that the inequality is satisfied. Then you might like to think about the conditions for which the inequality is actually an equality.

Only after a significant amount of play should we think about proving the statement.

For a purely symbolic proof of this special case of the triangle inequality, note that $|a||b| \geq 0$, and then reason as follows:

$$\begin{aligned}
0 &\leq |a||b| \\
0 &\leq 2|a||b| \\
|a|^2 + |b|^2 &\leq |a|^2 + |b|^2 + 2|a||b| \\
a^2 + b^2 &\leq (|a| + |b|)^2 \\
\sqrt{a^2 + b^2} &\leq |a| + |b|
\end{aligned}$$

Write a careful justification for each step in the proof. Then practice writing the proof until you can do it without looking at these steps. Focus on remembering the proof strategy, not on memorizing each step by rote; you'll retain the former a lot longer than the latter!

HISTORY

The history of Pythagoras's theorem

Pythagoras's theorem was known to Babylonian mathematicians at least 4000 years ago, to Chinese mathematicians at least 3000 years ago, and to Indian mathematicians at least but the theorem is named after Pythagoras because it is thought that he was the first to provide a proof. Pythagoras lived from 569 BCE to 500 BCE. It is also thought that Pythagoras used a proof similar to the geometric one provided earlier in this section, although this is also unclear.

There are old stories that builders used to use ropes with knots spaced by 3 units, 4 units, and 5 units, to construct triangles with right angles, so that they could ensure that the corners where walls met were square. These stories date back to the time of the building of the Egyptian pyramids, more than 4000 years ago, although there is absolutely no extant documentary evidence that the ancient Egyptians knew the theorem or used such rope triangles.

One of the most famous textbooks in history is *The Elements*, compiled by Euclid in Alexandria, Egypt, in about 300 BCE. Euclid took what was then known in mathematics and compiled it logically, starting with definitions and postulates, and then proceeding to state and prove theorems and describe various geometric constructions. The logical structure of the book influenced millennia of mathematicians, and was considered an essential part of general education into the 20th century, at which time its content was spread through many textbooks, and so it was no longer considered essential to read. Euclid includes a proof of Pythagoras's theorem, and is careful to state and prove the converse of the theorem as well.

There have been many different proofs of Pythagoras's theorem published through the ages, so much so that there is a whole genre of mathematical literature devoted to them. The role of Euclid's Elements in helping students learn logic means that it was a revered book, and stimulated much discussion about the various proofs in it, and alternative proofs. United States President James Garfield (1831–1881) discovered a new proof of Pythagoras's theorem in 1876. He had not yet become president, but was a member of the House of Representatives at the time. "He hit upon the proof in a mathematics discussion with some other members of Congress, and the proof was subsequently printed in the *New England Journal of Education*." (Source: *An Introduction to the History of Mathematics*, Fourth Edition, by Howard Eves, Holt, Rinehart, and Winston, 1976; page 128.) How times have changed.

Pythagoras's theorem can be generalized in various ways. One generalization applies to arbitrary triangles; this generalization is known as the cosine law: $c^2 = a^2 + b^2 - 2ab\cos C$, where C is the measure of the angle opposite the side with length c. You can check for yourself that if $C = 90°$ the cosine law reduces to Pythagoras's theorem.

Another way to generalize Pythagoras's theorem leads to Fermat's "last" theorem. Fermat wondered whether the equation $a^n + b^n = c^n$ could be satisfied by natural numbers a, b, c for exponents n that were different from 2. For example, are there three natural numbers

a, b, c that satisfy the equation $a^3 + b^3 = c^3$? The answer is no, and Fermat claimed that the answer is negative for all natural numbers n. He claimed to have a proof, but curiously did not publish it. It is very likely that he did not have such a proof, for although incremental progress was made over the following centuries, it was not until 1993 that Andrew Wiles famously solved this long-standing problems and published a proof, which relied on combining tools from various areas of mathematics that were unknown in the time of Fermat!

But that is a story for another time.

EXERCISES

(Answers at end.)

1. Determine the length of the hypotenuse of a right triangle if the lengths of the two other sides of the triangle are 2.31 m and 2.87 m.
2. Determine the length of one side of a right triangle if the length of the hypotenuse is 9.67 mm and the length of the other side is 5.29 mm.

Answers: 1. 3.68 m 2. 8.09 mm

2 A formula for the distance between two points

Using Pythagoras's theorem, we can develop a useful formula for the distance between two points in terms of the rectangular coördinates of the two points. First let's look at an example; consider the points $P(1,1)$ and $Q(4,5)$ plotted in Figure 7. In Figure 8, the same points are plotted, but in addition a line segment is drawn from P parallel to the x-axis and a line segment from Q is drawn parallel to the y-axis so that the two new line segments intersect at R. Because of the way the new line segments PR and QR were constructed, they are perpendicular, and so $\angle PRQ = 90°$. This means that we can apply Pythagoras's theorem to the right triangle PQR to determine the length of the hypotenuse PQ, which is the distance we are interested in determining.

Note that the length of PR is $a = 5 - 2 = 3$, and the length of QR is $b = 5 - 1 = 4$. Thus, using Pythagoras's theorem,

$$\begin{aligned} c^2 &= a^2 + b^2 \\ c^2 &= (5-2)^2 + (5-1)^2 \\ c^2 &= 3^2 + 4^2 \\ c^2 &= 9 + 16 \\ c^2 &= 25 \\ c &= 5 \end{aligned}$$

Thus, the distance between the points P and Q is 5 units.

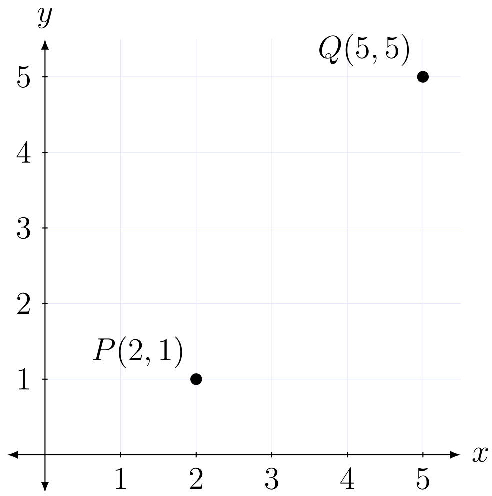

Figure 7: We would like to determine a formula for the distance between two points in terms of their coördinates. As a first step, we'll determine the distance between the specific points P and Q indicated in the diagram.

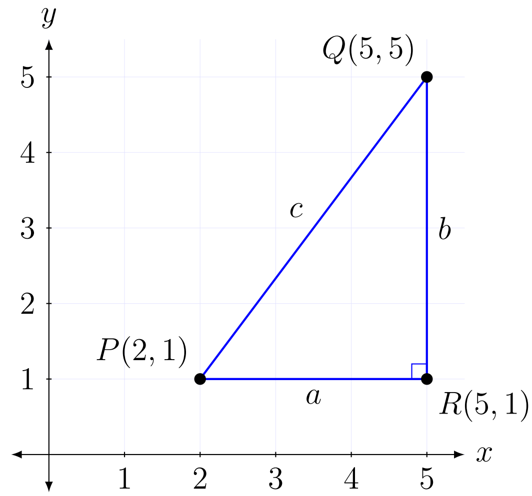

Figure 8: By constructing the right triangle in the figure, we see that we can use Pythagoras's theorem to determine the distance between the points P and Q.

Now let's repeat the same argument in general; that is, suppose that the rectangular coördinates of points P and Q are $P(x_1, y_1)$ and $Q(x_2, y_2)$, as in Figure 9.

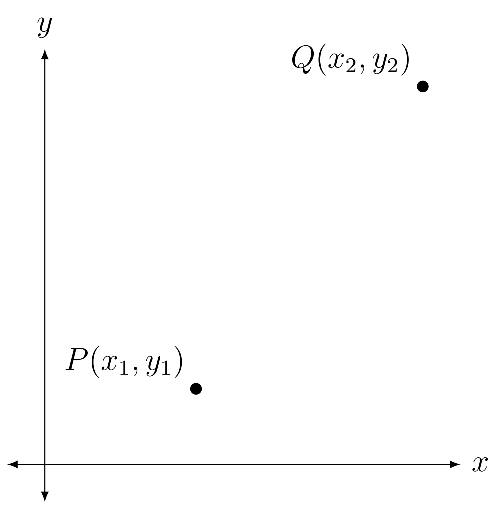

Figure 9: The text develops a formula for the distance between two points in terms of their coördinates (x_1, y_1) and (x_2, y_2).

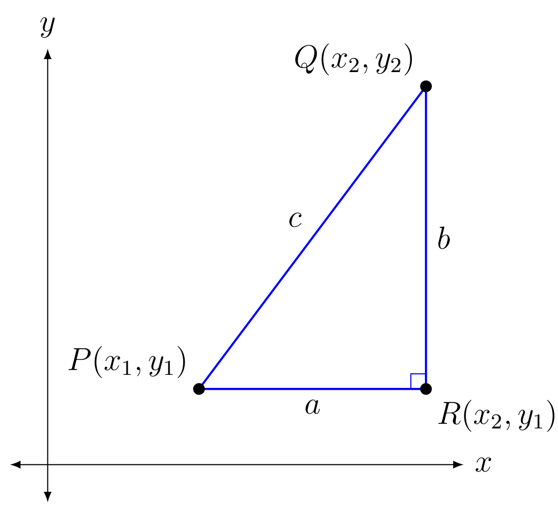

Figure 10: By constructing the right triangle in the figure, we see that we can use Pythagoras's theorem to develop a formula for the distance between the points P and Q in terms of their coördinates, as shown in the text.

Then the distance between the points P and Q is c, where

$$\begin{aligned} c^2 &= a^2 + b^2 \\ c^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ c &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned}$$

KEY CONCEPT

Formula for the distance between two points in terms of their rectangular coördinates

The distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

It is not necessary to memorize this formula; it is better to practice using it, and to remember the idea behind the formula, which is nothing else but Pythagoras's theorem. Pythagoras's theorem is more fundamental, and it is always better to remember the fundamentals and practice deriving consequences from them, as this will improve your long-term retention.

EXAMPLE 3

Using the formula for the distance between two points

Determine the distance between the points that have coördinates $(-1, 3)$ and $(2, -5)$.

SOLUTION

Using the distance formula, we can calculate the distance between the two points as follows.

$$\begin{aligned} \text{distance} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ \text{distance} &= \sqrt{(2 - [-1])^2 + (-5 - 3)^2} \\ \text{distance} &= \sqrt{(3)^2 + (-8)^2} \\ \text{distance} &= \sqrt{9 + 64} \\ \text{distance} &= \sqrt{73} \\ \text{distance} &\approx 8.54 \end{aligned}$$

Does the result seem reasonable? Can you check it in some independent way? Sketching a graph may help you to decide whether the result is reasonable.

EXERCISES

(Answers at end.)

Determine the distance between each pair of points.

1. Calculate the distance between the point $(2, -1)$ and the point $(3, 2)$.
2. Calculate the distance between the point $(-2, -3)$ and the point $(4, 1)$.
3. Calculate the distance between the point $(-3, -5)$ and the point $(5, -5)$.
4. Calculate the distance between the point $(-4, -2)$ and the point $(-5, 5)$.

Answers: 1. $\sqrt{10}$ 2. $2\sqrt{13}$ 3. 8 4. $\sqrt{50}$

3 The midpoint of a line segment

It's possible to derive a formula for the rectangular coördinates of the midpoint of a line segment in terms of the coördinates of the endpoints. We can accomplish this in two different ways, reasoning based on similar triangles or using the formula for the distance between two points. We'll show both methods. First consider a specific example; let's determine the midpoint of the line segment PQ, where the coördinates are $P(2, 1)$ and $Q(5, 5)$.

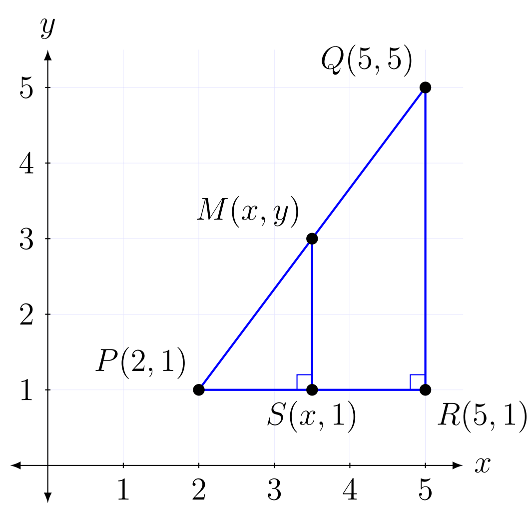

Figure 11: We would like to determine the coördinates of the midpoint of the line segment PQ. To do so, we can construct the similar right triangles shown in the figure, and argue using the properties of similar triangles, as shown in the text.

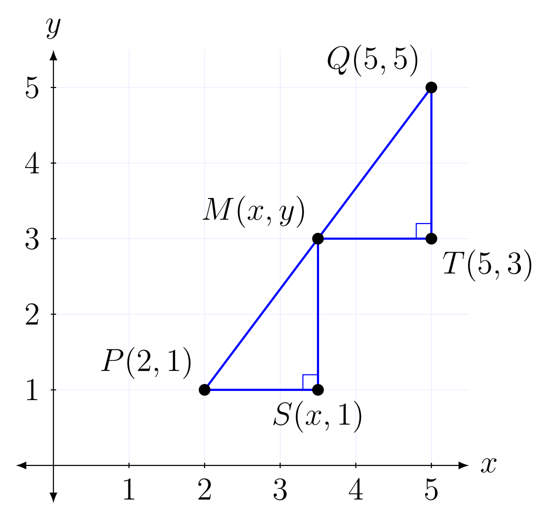

Figure 12: An alternative method for determining the coördinates of the midpoint of the line segment PQ is to construct the right triangles shown in the figure, and argue using the distance formula, as shown in the text.

To determine the coördinates of the point M, the midpoint of the line segment PQ, using similar triangles, first we must show that triangles ΔMPS and ΔQPR are similar. In Figure

11, the corresponding angles $\angle MSP$ and $\angle QRP$ are both right angles by construction, and $\angle MPS$ is the same as $\angle QPR$. The third pair of corresponding angles must therefore also be the same, because the sum of the angles in a triangle is 180°. Thus, triangles ΔMPS and ΔQPR are similar.

Because we have constructed M to be the midpoint of line segment PQ,

$$PM = \frac{1}{2}PQ$$

Corresponding sides of the similar triangles are in the same ratio, so therefore

$$PS = \frac{1}{2}PR \qquad \text{and} \qquad MS = \frac{1}{2}QR$$

which means that

$$x - 2 = \frac{1}{2}(5 - 2) \qquad \text{and} \qquad y - 1 = \frac{1}{2}(5 - 1)$$

and therefore

$$\begin{aligned} x &= 2 + \frac{3}{2} & \qquad\qquad y &= 1 + \frac{4}{2} \\ x &= \frac{4}{2} + \frac{3}{2} & y &= 1 + 2 \\ x &= \frac{7}{2} & y &= 3 \\ x &= 3.5 & y &= 3 \end{aligned}$$

Thus, the coördinates (x, y) of the midpoint M of the line segment QP are $M(3.5, 3)$.

Alternatively, we can use an argument based on the distance formula to determine the coördinates of the midpoint M of the line segment PQ. In Figure 12, the definition of midpoint is that the distance from P to M is equal to the distance from M to Q. Therefore, the squares of the distances are also equal, and so

$$(x - 2)^2 + (y - 1)^2 = (5 - x)^2 + (5 - y)^2$$

But hold on — before we go any further, we should notice that the previous equation is one equation with two unknowns, so we don't have any hope of solving it without additional information. We need a second independent equation. But what is going on geometrically? Have we made a mistake, or is there some reason why the condition we have specified does not pin down the point M definitively?

The condition we specified in writing the previous equation is that the point M is equally distant from the points P and Q. Are there other points that are equally distant from the points P and Q? You might like to think about this and play with a diagram before reading on.

The answer to the question is, "Yes." Every point on the perpendicular bisector of PQ, which is the dashed line in Figure 13, is equally distant from P and Q. So this condition alone is not enough to uniquely define the point M; we need another condition. But what is special about M, among all of the points on the perpendicular bisector of PQ? It is the only point on the dashed line that is also on the line segment PQ. OK, so we have to include a condition that specifies that M is on the line segment PQ. Another way to say this is that the slope of the line segment PM is the same as the slope of the line segment PQ; expressing this in symbols will give us a second, independent equation involving x and y:

$$\begin{aligned}
\text{slope of } PM &= \text{slope of } PQ \\
\frac{y-1}{x-2} &= \frac{5-1}{5-2} \\
\frac{y-1}{x-2} &= \frac{4}{3} \\
y-1 &= \left(\frac{4}{3}\right)(x-2) \\
y &= 1 + \left(\frac{4}{3}\right)(x-2)
\end{aligned}$$

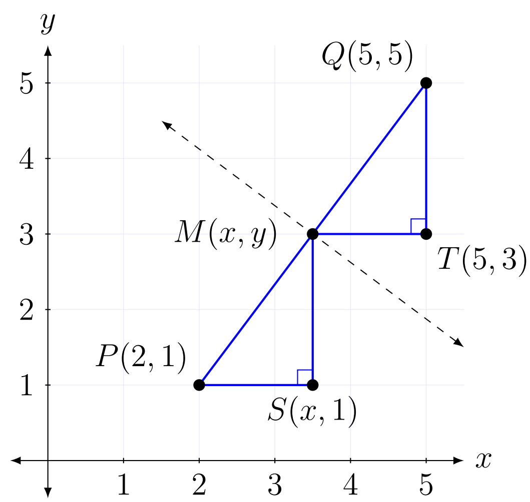

Figure 13: Every point on the perpendicular bisector of the line segment PQ is equally distant from the points P and Q.

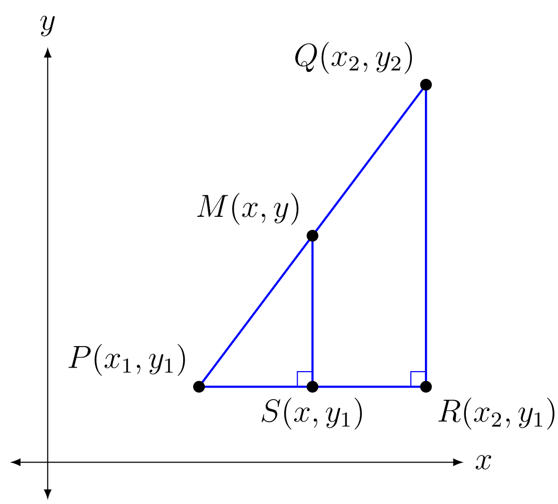

Figure 14: This figure is used in determining a general formula for the midpoint of a line segment in terms of the coördinates of the endpoints.

Substituting the expression for y in the previous equation in the equation that relates the distances PM and MQ,

$$(x-2)^2 + (y-1)^2 = (5-x)^2 + (5-y)^2$$

results in

$$(x-2)^2 + \left(\left[1 + \left(\frac{4}{3}\right)(x-2)\right] - 1\right)^2 = (5-x)^2 + \left(5 - \left[1 + \left(\frac{4}{3}\right)(x-2)\right]\right)^2$$

At some point, perhaps now when we are facing the previous equation, which is quite messy, we shall have realized that the previous method (using similar triangles) is far better than this one. But this is the best way to learn: Try different methods, play with the material, make lots of mistakes, take many paths that are not the best, and thereby learn which paths are the best. For the intrepid among you, who would like to continue along this path, I shall complete this calculation so that you can check your work against it.

$$
\begin{aligned}
(x-2)^2 + \left(\frac{4}{3}\right)^2 (x-2)^2 &= (5-x)^2 + \left(4 - \left(\frac{4}{3}\right)(x-2)\right)^2 \\
(x-2)^2 + \frac{16}{9}(x-2)^2 &= (5-x)^2 + \left(16 - \left(\frac{32}{3}\right)(x-2) + \frac{16}{9}(x-2)^2\right) \\
(x-2)^2 &= (5-x)^2 + 16 - \left(\frac{32}{3}\right)(x-2) \\
x^2 - 4x + 4 &= 25 - 10x + x^2 + 16 - \left(\frac{32}{3}\right)(x-2) \\
0 &= 37 - 6x - \left(\frac{32}{3}\right)(x-2) \\
0 &= 111 - 18x - 32(x-2) \\
0 &= 111 - 18x - 32x + 64 \\
50x &= 175 \\
x &= \frac{175}{50} \\
x &= 3.5
\end{aligned}
$$

Finally we substitute this value of x into the following relation between x and y

$$
y = 1 + \left(\frac{4}{3}\right)(x-2)
$$

to obtain the y-coördinate of the midpoint:

$$
\begin{aligned}
y &= 1 + \left(\frac{4}{3}\right)(3.5-2) \\
y &= 1 + \left(\frac{4}{3}\right)(1.5) \\
y &= 1 + \left(\frac{4}{3}\right)\left(\frac{3}{2}\right) \\
y &= 1 + 2 \\
y &= 3
\end{aligned}
$$

We obtain the same coördinates of the midpoint M as before, $(3.5, 3)$. To repeat, the calculation based on similar triangles is much simpler and therefore preferred. But it's always a good check on your work to have two independent methods for calculating some quantity.

Now let's repeat the calculation based on similar triangles in a general situation, to obtain a formula. Consider Figure 14 for the notation. Using the similar triangles ΔMPS and ΔQPR, the fact that

$$
PM = \frac{1}{2} PQ
$$

implies that

$$PS = \frac{1}{2} PR \qquad \text{and} \qquad MS = \frac{1}{2} QR$$

which means that

$$x - x_1 = \frac{1}{2}\left(x_2 - x_1\right) \qquad \text{and} \qquad y - y_1 = \frac{1}{2}\left(y_2 - y_1\right)$$

and therefore

$$\begin{aligned} x &= x_1 + \frac{1}{2}\left(x_2 - x_1\right) & y &= y_1 + \frac{1}{2}\left(y_2 - y_1\right) \\ x &= \frac{2}{2}x_1 + \frac{1}{2}\left(x_2 - x_1\right) & y &= \frac{2}{2}y_1 + \frac{1}{2}\left(y_2 - y_1\right) \\ x &= \frac{1}{2}\left(x_2 - x_1 + 2x_1\right) & y &= \frac{1}{2}\left(y_2 - y_1 + 2y_1\right) \\ x &= \frac{1}{2}\left(x_2 + x_1\right) & y &= \frac{1}{2}\left(y_2 + y_1\right) \end{aligned}$$

KEY CONCEPT

Formula for the midpoint of a line segment

If the coördinates of P and Q are $P(x_1, y_1)$ and $Q(x_2, y_2)$, then the coördinates (x, y) of the midpoint M of the line segment PQ are

$$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

In words, the x-coördinate of the midpoint is the average of the x-coördinates of the endpoints, and the y-coördinate of the midpoint is the average of the y-coördinates of the endpoints.

CAREFUL!

Ambiguous notation

There are many mathematical concepts, and only a limited number of symbols, which means that it's inevitable that one symbol will stand for a number of different concepts. There is a clear disadvantage to this, because of the potential for confusion or unnoticed misunderstanding — that which is intended might be misconstrued as something else. However, there is the potential for economy of expression, especially once one has understood a concept, in that the same symbol can stand for a number of related concepts, depending on context, allowing for efficient exposition.

In this section we have used symbols such as PQ to represent both a line segment connecting points P and Q, and the length of the line segment. Does it bother you that we have used the same notation to mean two different (although closely related) things? If so, how would you improve the notation? After you have given this some though, browsing through some other resources will give you a sense for what other authors do.

EXAMPLE 4

Using the formula for the midpoint of a line segment

Determine the coördinates of the midpoint of the line segment that has endpoints $(5, -3)$ and $(1, 7)$.

SOLUTION

The coördinates of the midpoint are

$$\begin{aligned}
(x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\
(x, y) &= \left(\frac{5 + 1}{2}, \frac{-3 + 7}{2}\right) \\
(x, y) &= \left(\frac{6}{2}, \frac{4}{2}\right) \\
(x, y) &= (3, 2)
\end{aligned}$$

The coördinates of the midpoint are $(3, 2)$. Does this seem reasonable? Is it possible to check this in some independent way? Can sketching a graph be helpful?

GOOD QUESTION

Separating line segments into parts that have a given ratio

We have learned a formula for the coördinates of the midpoint of a line segment in terms of the coördinates of the endpoints of the line segment. The midpoint separates the line segment into two parts of equal length. I wonder if there is a nice formula that allows you to determine the coördinates of a point on the line segment that separates it into two parts, one of which is twice as long as the other? Or three times as long? Or k times as long, where k is any numerical factor?

Can you determine such a formula? Can you prove that it is correct?

EXERCISES

(Answers at end.)

1. Determine the coördinates of the midpoint of the line segment that has endpoints $(1, 2)$ and $(-1, -4)$.

2. Determine the coördinates of the midpoint of the line segment that has endpoints $(-3, -6)$ and $(5, 8)$.

Answers: 1. $(0, -1)$ 2. $(1, 1)$

4 A summary of useful mensuration formulas

Most of these formulas are learned in high school, although I believe it is not common to learn Heron's formula in school nowadays. You might like to check into Heron's formula (and any of the others that you would like to explore) to learn how it is derived. These formulas come up frequently, and it will be helpful for you to internalize them.

circumference C of a circle of radius r: $C = 2\pi r$

area A of a circle of radius r: $A = \pi r^2$

surface area S of a sphere of radius r: $S = 4\pi r^2$

volume V of a sphere of radius r: $V = \frac{4}{3}\pi r^3$

area A of a rectangle of length L and width W: $A = LW$

area A of a triangle of height h and base b: $A = \frac{1}{2}bh$

area of a trapezoid: $\left(\dfrac{a+b}{2}\right)h$, where a and b are the lengths of the parallel edges and h is the height of the trapezoid (measured perpendicular to the parallel edges)

Heron's formula for the area of a triangle with side lengths a, b, and c: $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \dfrac{1}{2}(a+b+c)$

For the latest version of these notes, see:

`https://fomap.org/prepare-for-university/linear-algebra-part-1/`