Linear Algebra, Part 1

Lecture 2-1: Cartesian Coördinates

1 Coördinate systems

OVERVIEW

The idea of using "numerical addresses" to identify points in space was a tremendous advance in mathematics, as it facilitated the use of algebraic and analytical tools in geometry. Geometry was considered a separate realm from analysis in the old days, and the fusion of the two beginning in the 1500s provided the foundation for modern mathematics, which spurred a revolution in science and in our understanding of the universe. Fusing geometry and algebra to help us understand the fundamental concepts of linear algebra is one of the themes of this book.

In this chapter we begin this study by developing some of the basic tools of analytic geometry, using numbers and formulas to describe and provide insight into geometric concepts. The connections between geometry and algebra are mediated by coördinate systems. A coördinate system is a system for labelling each point in a geometrical situation using numbers. The simplest coördinate systems are called rectangular coördinate systems, or Cartesian coördinate systems, in honour of the French mathematician René Descartes.

There are many other types of coördinate systems that are useful for various purposes. For example, pilots use headings, which are like polar coördinates (discussed later in this book), latitude and longitude are used to specify points on the surface of the Earth, which are somewhat like spherical coördinates, and so on. These and other useful coördinate systems are discussed in higher mathematics courses.

In applications, it's often greatly helpful to choose a coördinate system that is adapted to the symmetry that is present in the physical system that is being studied. In advanced work, many coördinate systems are studied; in this section, we begin by studying the basics.

The idea of using a coördinate system is to describe each location in space using numbers. This allows us to discuss relations among the coördinates of points that belong to geometrical figures in terms of equations that the coördinates satisfy, and in this way we can use analytic tools to study geometry. On the other hand, we can also ask about the geometry that corresponds to various formulas or the solutions of equations if we interpret the variables in terms of coördinates. This allows us to bring geometric perspectives and physical insight into the realm of formulas and solving equations. By connecting geometry and analysis, we enrich both, and aid our understanding of both.

There are different types of coördinate systems, and it is wise to use the one that is most helpful in a given situation. Typically we search for any symmetry present in a situation, and then choose a coördinate system that conforms with the symmetry. If there is no geometric symmetry, then we might examine the formulas present and choose a coördinate system that makes the algebraic work easier. In any case, having a number of basic coördinate systems at your command will allow you to choose the appropriate one in practice.

For one-dimensional problems (such as a car driving along a straight highway), which can therefore be modelled by a number line, one can specify each point on the line using a single number. For two-dimensional problems (such as a person walking in a flat field), one needs a pair of numbers. The most commonly used coördinate systems for (two-dimensional) planes are rectangular (Cartesian) coördinates and polar coördinates, although there are infinitely many alternatives. For (flat) three-dimensional spaces, the most common coördinate systems are rectangular (Cartesian) coördinates, spherical polar coördinates, and cylindrical polar coördinates. On the earth's surface (which we approximate as the two-dimensional surface of a sphere), a variant of spherical coördinates is used, where each point is specified by two angles, known as latitude and longitude.

The idea of using a coördinate system is somewhat like the idea of using addresses in daily life. Identifying each building in a city using an address makes it easier to find a specific building when you need to do so. However, the systems used for addresses in cities are not quite as systematic as the ones used in mathematics; as usual, mathematicians prefer simplicity, precision, and no ambiguity. Streets in cities are sometimes numbered, sometimes named, and sometimes the name of a street changes several times along it. There are conventions for numbering buildings on streets; some cities use even numbers for the North and West sides, odd numbers for the South and East sides, but what do you do when a street curves in a complicated way? Conventions must be used, and special cases abound. Sometimes numbers increase by 4 for each building, sometimes by 2, sometimes by other numbers. Sometimes the street address gives you insight into which cross-streets are nearby, sometimes not. Street addresses are almost always whole numbers. Not every position in a city has a street address, only buildings.

In contrast, the position of a point in a plane is uniquely specified by its Cartesian coördinates. That is, every point in the plane has coördinates, and the coördinates of each point are different from the coördinates of every other point. This uniqueness is important: Because each point has its own coördinates, different from every other point, as soon as we are told the coördinates of a point, we know exactly where it is.

HISTORY

The rise of algebra beginning in the 1500s

Up to the late 1500s algebra and geometry were for the most part distinct areas of study with little overlap. Simple equations resulting from practical geometric problems had been solved up to then, but algebraic methods were not used in geometrical investigations until the 1600s, with the work of Pierre de Fermat (1601–1665), although Rene Descartes (1596–1650) often erroneously gets the credit. Fermat bridged algebra and geometry by noting that geometrical figures can be represented by equations if a coördinate system is introduced. Now algebra could be used as a powerful tool to discover and prove geometrical properties. Conversely, equations could be interpreted geometrically, providing a more concrete understanding of their properties. The study of geometry using algebraic techniques is nowadays called analytic geometry or coördinate geometry. This is an important foundation for further studies in mathematics, including calculus.

However, Fermat and his contemporaries did not accept negative numbers, so their studies were handicapped. Their successors used and accepted negative numbers as reasonable and so further developed analytic geometry. In a parallel development, Galileo Galilei (1564–1642) pioneered the use of mathematical formulas to describe motion, in contrast to earlier, vaguer descriptions of motion that were wedded to speculations about why objects moved. Since information about motions could be deduced so readily using algebraic techniques, algebra grew in stature and became more widely accepted. Nevertheless, algebra was considered for a long time to be subordinate to geometry—although their usefulness was widely accepted, algebraic methods were heavily criticized for lacking logical foundations and for the absence of rigorous proofs.

2 Rectangular Coördinates

In this section we'll discuss rectangular coördinates in the plane. The idea is to use two numbers to specify the location of each point in the plane. To do this, we begin by drawing two reference lines, typically called the x-axis and the y-axis, at right angles to each other, and intersecting at a point that we call the origin of the coördinate system. We typically place the reference lines so that the x-axis is "horizontal" as drawn on a page, and the y-axis is "vertical" as drawn on a page. See Figure 1.

A number of points are plotted in Figure 2 and are labelled with their rectangular coördinates. Note that the origin of the coördinate system, which is the point at which the x-axis and y-axis intersect, has coördinates $(0, 0)$. In any pair of coördinates, the first coördinate is called the x-coördinate and the second coördinate is called the y-coördinate. Consider the point in Figure 2 that has coördinates $(1, 3)$. The meaning of the coördinates is that starting from the origin, we could reach the indicated point by first moving 1 unit to the right along the x-axis and then moving 3 units upwards parallel to the y-axis. To reach the point $(-2, 1)$ starting from the origin, first move 2 units to the left along the x-axis, and then move 1 unit upward parallel to the y-axis.

If several points are plotted in the same diagram, as in Figure 2, we often include an additional label, such as a letter, in order to make it easier to talk about them. For example, we might label the points in Figure 2 as $A(1, 3)$, $B(-2, 1)$, $C(2.73, -1.81)$, and $D(-\sqrt{3}, -\sqrt{2})$. If we later referred to the point D, then it would be clear that we mean the one with coördinates $(-\sqrt{3}, -\sqrt{2})$.

The coördinate axes divide the plane into four regions, called quadrants. In the upper-right quadrant, also called the first quadrant, both of the coördinates are positive numbers. In the upper-left quadrant, also called the second quadrant, the x-coördinate is negative and the y-coördinate is positive. In the lower-left quadrant, also called the third quadrant, both coördinates are negative. Finally, in the lower-right quadrant, also called the fourth quadrant, the x-coördinate is positive and the y-coördinate is negative.

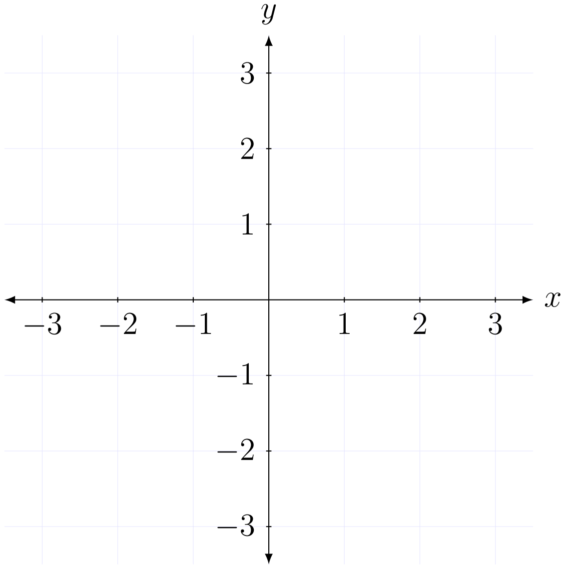


Figure 1: The figure shows a rectangular (Cartesian) coördinate system. The reference lines are the x-axis (which is "horizontal") and the y-axis (which is "vertical"). The reference lines intersect at a right angle, and their intersection point, which has coördinates $(0, 0)$, is called the origin.

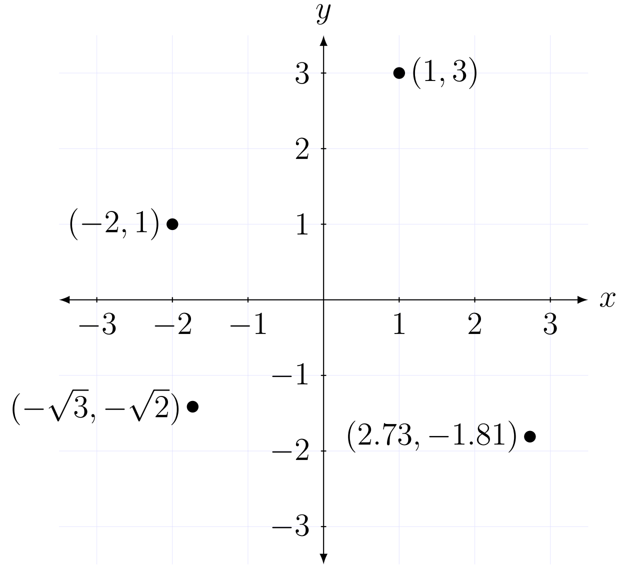


Figure 2: Some sample points plotted along with their coördinates. Note that each point in the plane has its own unique pair of coördinates, and that the coördinates are real numbers (that is, the coördinates do not have to be whole numbers).

All points that are on the x-axis have y-coördinate equal to 0, so the coördinates of a point on the x-axis are of the form $(x, 0)$. All points that are on the y-axis have x-coördinate equal to 0, so the coördinates of a point on the y-axis are of the form $(0, y)$.

Also note that we often pretend that a figure with a rectangular coördinate system represents a landscape, and so we often interpret the y-coördinate of a point as a height. If the y-coördinate is positive, we often speak of a height above the x-axis, and if the y-coördinate is negative, we often speak of a depth below the x-axis.

MAKING CONNECTIONS

Algebraic notation in chess

In the old days, chess moves were indicated in a notation such as Kt-KB3, meaning a knight had moved to the square that is third from one side of the board along the column where the king's bishop begins play. However, this is ambiguous, because the same notation would be used to indicate a black knight moving to a square third from black's side of the board and a white knight moving to a different square, one that is third from white's side of the board.

Nowadays, virtually all chess publications use what is called “algebraic notation,” which is analogous to using rectangular coördinates in mathematics. The bottom-left corner of the board from white’s perspective is considered the origin, and the square nearest this corner is labelled $a1$. The columns of the board (called files in chess) are labelled by letters a to h inclusive, and the rows of the board (called ranks in chess) are labelled by numbers 1 to 8 inclusive, relative to the origin. Thus, each square on the board has a unique address, just as in the rectangular coördinate system. (However, there are only 64 possible addresses on the chess board, and an infinite number of possible sets of coördinates on a mathematical plane.)

In algebraic notation, the moves given earlier would be indicated Nf3 for the white move, and Nf6 for the black move. (Nowadays the letter N is used to indicate knight rather than Kt.) Many modern chess books use “figurine algebraic notation,” where the pieces are denoted by a small figure instead of a letter; thus, the moves given earlier would be typeset as ♘f3 and ♘f6. This, along with the international convention for uniquely naming squares, allows games to be printed and replayed by anyone in the world, without having to know the language of the person who recorded the game.

$a8$	$b8$	$c8$	$d8$	$e8$	$f8$	$g8$	$h8$
$a7$	$b7$	$c7$	$d7$	$e7$	$f7$	$g7$	$h7$
$a6$	$b6$	$c6$	$d6$	$e6$	$f6$	$g6$	$h6$
$a5$	$b5$	$c5$	$d5$	$e5$	$f5$	$g5$	$h5$
$a4$	$b4$	$c4$	$d4$	$e4$	$f4$	$g4$	$h4$
$a3$	$b3$	$c3$	$d3$	$e3$	$f3$	$g3$	$h3$
$a2$	$b2$	$c2$	$d2$	$e2$	$f2$	$g2$	$h2$
$a1$	$b1$	$c1$	$d1$	$e1$	$f1$	$g1$	$h1$

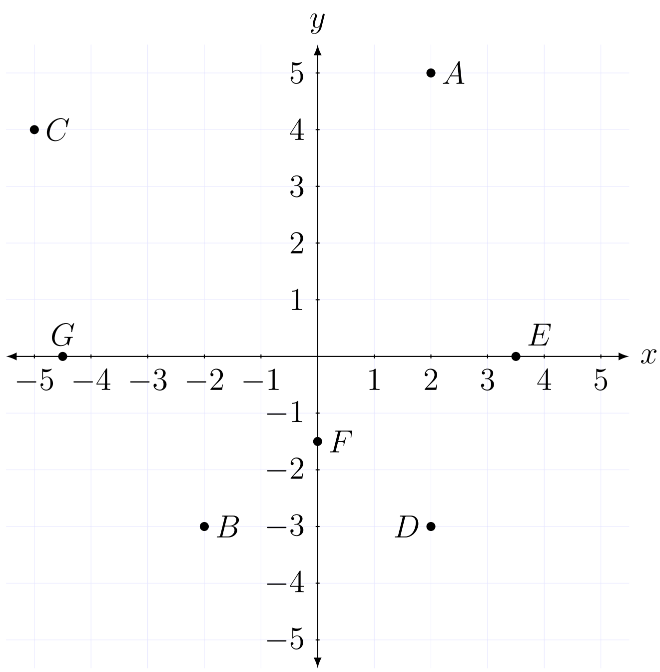
Figure 3: The figure shows a chessboard, from white’s perspective, with the squares uniquely labelled, as is done nowadays, using algebraic notation. This chess notation is analogous to using rectangular coördinates in a plane in mathematics.

EXERCISES

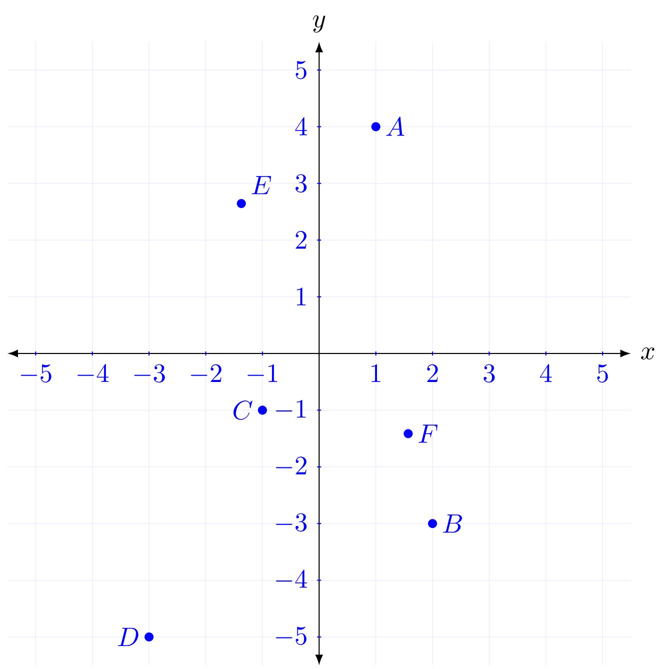
(Answers at end.)

1. Plot the following points using a rectangular (Cartesian) coördinate system: $A(1,4)$, $B(2,-3)$, $C(-1,-1)$, $D(-3,-5)$, $E(-1.372, 2.645)$, and $F(\pi/2, -\sqrt{2})$.

2. Identify the coördinates of the points labelled in Figure 3.



Answers: 1.



2. (a) $A(2,5)$ (b) $B(-2,-3)$ (c) $(-5,-4)$ (d) $(2,-3)$ (e) $(3.5,0)$ (f) $(0,-1.5)$ (g) $(-4.5,0)$

HISTORY

History of the development of coördinate systems

The idea of using coördinates to identify points in space has its origin with the ancient Greeks, although they lacked the idea of an equation relating variable quantities. Oresme, in the 1300s, did indeed begin to play with the idea of representing graphically a relationship between two variable quantities. For example, he plotted what we would nowadays call velocity-time graphs, with the time variable plotted along a horizontal axis (he called the various plotted times longitudes on the diagram) and then line segments plotted perpendicular to the horizontal axis representing the velocities at the various times (he called the lengths of the plotted vertical line segments latitudes). His primary interest was the area of the resulting geometrical figure, and so the further development of this idea of representing a functional relationship by a curve on a plane with a coördinate system had to wait until the 1600s. Notable developments were made by Fermat and Descartes, and the actual word "coördinate" was introduced, along with some other standard terminology, by Leibniz in 1692.

By the 1600s, the need for applying mathematics to science had grown far beyond Euclidean geometry, which was then the core of mathematics. Euclidean geometry dealt primarily with lines and circles, and new scientific discoveries in the 1600s in astronomy, optics, and motion required an understanding of many different kinds of curves. The methods of Euclidean geometry were insufficient, and this motivated new ways to understand curves quantitatively.

In an essay that Fermat wrote in 1629 and circulated to the French mathematical community in early 1637, he introduced perpendicular coördinate axes; found general equations for straight lines; circles, ellipses, parabolas, and hyperbolas; and showed that every first or second degree equation corresponds to a curve of one of these types. Descartes published his great work on analytic geometry, *La géométrie* in the same year 1637, although he had worked out many of his ideas about ten years earlier. Although Fermat's work was more systematic and perhaps more instructive, because it was never published in his lifetime Descartes's work was more influential. And what an influence; calculus was being developed by Fermat and many other contemporaries (culminating in the systematizing work of Newton and Leibniz in the late 1600s), and relied on the firm foundation of analytic geometry.

It is amazing that Fermat was able to become one of the greatest mathematicians in history as an amateur! Fermat was trained as a lawyer, and he worked full-time as a lawyer and later as a councillor in his local parliament. He made his great mathematical discoveries while relaxing in the evenings! Descartes, on the other hand, was more interested in philosophy and science than in mathematics, and indeed his great work *A Discourse on Method* was very influential with its emphasis on reason in science independent of a reliance on theological scriptures.

Sources for further reading:

A History of Mathematics, by Carl B. Boyer, Wiley, 1968.

Mathematics in Western Culture, by Morris Kline, Oxford University Press, 1964.

For the latest version of these notes, see:

`https://fomap.org/prepare-for-university/linear-algebra-part-1/`