Linear Algebra, Part 1

Tutorial 2-4: Parametric Equations

In this tutorial you will how to describe lines using what are called parametric equations. There are several benefits to this form of description, including the ability to model physical motions along lines, a way of generalizing the equation of a line to three-dimensional space, an idea for describing planes and other surfaces in three-dimensional space, a deeper understanding of the idea of dimension, and a convenient way to express the solution to a system of equations (as we shall learn in the following chapter). Answers to the exercises are found at the end of the document.

1. Consider the line that has formula $y = 2x + 1$. Imagine a particle that is moving along the line at a constant speed. The particle is at the point $(-3, -5)$ when the timer is at $t = 0$ and the particle is at the point $(1, 3)$ when the timer is at $t = 2$ s.

   (a) Where is the particle when $t = 1$ s?

   (b) Where is the particle when $t = 3$ s?

   (c) Where is the particle when $t = 4$ s?

   (d) Determine a formula for the x-coördinate of the particle's position in terms of time. One way to do this is to start with a table of values with t-values in the first column and x-values in the second column, and then take it from there.

   (e) Determine a formula for the y-coördinate of the particle's position in terms of time.

   The formulas for x and y in terms of t in Parts (d) and (e) of the previous exercise are called parametric equations of the line being considered.

2. Determine the standard formula for the line that has the given parametric equations. One way to do this is by solving one of the parametric equations for t and then substituting the resulting formula for t into the other parametric equation. Another way is to solve each of the parametric equations for t and equate the resulting formulas. (In each case, t is eliminated, and you are left with a relation between x and y.)

   (a) $x = t + 1$ and $y = 2t + 3$

   (b) $x = 2t - 5$ and $y = 4t - 9$

   (c) $x = -t - 1$ and $y = -2t - 1$

   (d) $x = -2t + 2$ and $y = -4t + 5$

3. The point of the previous exercise is that one particular line can have many different sets of parametric equations. In fact, there are an infinite number of different sets of parametric equations for a given line. Physically, does this make sense? This allows us to model an infinite number of different motions that take place along the line; each set of parametric equations describes a different motion along the same line.

   In each part of the previous exercise, where is the particle at $t = 0$? In which direction does the particle move? What is the speed of the particle?

4. Despite the difference in physical meaning for different sets of parametric equations that describe the same line, sometimes we only need one set of parametric equations, and we don't care much about any physical interpretation. Therefore, obtaining a set of parametric equations, *any set*, is a valuable skill.

   A straightforward way to determine one set of parametric equations for a line that passes through the points A and B is to, nevertheless, think physically and assume that a particle moves along the line at a constant speed, reaching point A at time $t = 0$ and point B at time $t = 1$. (Of course, there are an infinite number of other ways, but this way is simple, and simple is often good, or at least quite sufficient.)

   Determine a set of parametric equations for the line that passes through each pair of given points.

   (a) $(1, 2)$ and $(3, 4)$
   (b) $(1, -1)$ and $(2, 5)$
   (c) $(-3, 2)$ and $(-5, 1)$
   (d) $(3, -3)$ and $(6, 2)$

5. Determine a set of parametric equations for the line that passes through the points (x_1, y_1) and (x_2, y_2).

6. So far we have considered particles moving on lines, and attending to their motion after a timer starts (i.e., after $t = 0$). However, you can imagine that the moving particle started indefinitely in the past and then will proceed indefinitely into the future, so that the particle will indeed be at each point on the line at a particular time in the past or future.

   Unless otherwise specified, often it is understood that the domain of the parameter (which we often consider to be time in physical contexts, but need not have any physical meaning in a purely mathematical context) is all real numbers.

   If a formula for a function is given, the simplest way to parameterize the resulting graph is to simply let $x = t$, and then substitute t for x in the given formula for the function to determine the parametric equation for y. (Of course, there are an infinite number of other possibilities, but if simplest works, then often simplest is best.)

   Determine a set of parametric equations for each function. Specify the domain of the parameter.

   (a) $y = x^2$

(b) $x = y^2$

(c) $y = \sqrt{x}$

(d) $y = -\sqrt{x}$

(e) $y = 2^x$

(f) $y = \log x$

7. Curves that are relations but not functions can also be described using parametric equations. (We have already seen one such example in the previous exercise.) Determine a formula (involving just x and y) for each relation that is described by the given set of parametric equations. Plot a graph of each relation. Describe the motion of a particle if x and y represent the coördinates of the particle's position at time t.

   (a) $x = t^2$ and $y = t$

   (b) $x = \cos(t)$ and $y = \sin(t)$

   (c) $x = \cos(2t)$ and $y = \sin(2t)$

   (d) $x = \cos(-t)$ and $y = \sin(-t)$

   (e) $x = 3\cos(2\pi t)$ and $y = 3\sin(2\pi t)$

   (f) $x = r\cos(2\pi ft)$ and $y = r\sin(2\pi ft)$, where $r > 0$

Answers to exercises:

1. (a) $(-1, -1)$ (b) $(3, 7)$ (c) $(5, 11)$ (d) $x = 2t - 3$ (e) $y = 4t - 5$

2. (a) $y = 2x + 1$ (b) $y = 2x + 1$ (c) $y = 2x + 1$ (d) $y = 2x + 1$

3. Because the particles travel at constant speed, you can determine their speeds by dividing the distance they travel (use Pythagoras's theorem!) by the time interval.

(a) At $t = 0$ the particle is at the point $(1, 3)$. The particle moves upwards and to the right along the line at a constant speed of $\sqrt{5}$ in unspecified units. (For example, if x and y are measured in metres, and t is measured in seconds, then the speed of the particle is $\sqrt{5}$ m/s.)

(b) At $t = 0$ the particle is at the point $(-5, -9)$. The particle moves upwards and to the right along the line at a constant speed of $2\sqrt{5}$ in unspecified units.

(c) At $t = 0$ the particle is at the point $(-1, -1)$. The particle moves downwards and to the left along the line at a constant speed of $\sqrt{5}$ in unspecified units.

(d) At $t = 0$ the particle is at the point $(2, 5)$. The particle moves downwards and to the left along the line at a constant speed of $2\sqrt{5}$ in unspecified units.

4. There are an infinite number of different sets of parametric equations in each case, so your sets might be quite valid, even if it is not the same as the sets listed here. The sets of parametric equations listed below are valid, but an infinite number of other sets are also valid.

(a) $x = 2t + 1$ and $y = 2t + 2$ (b) $x = t + 1$ and $y = 6t - 1$

(c) $x = -2t - 3$ and $y = -t + 2$ (d) $x = 3t + 3$ and $y = 5t - 3$

5. There are an infinite number of valid sets of parametric equations. Here is one:

$$x = (x_2 - x_1)\, t + x_1 \quad \text{and} \quad y = (y_2 - y_1)\, t + y_1$$

which is equivalent to

$$x = (1 - t)\, x_1 + t x_2 \quad \text{and} \quad y = (1 - t)\, y_1 + t y_2$$

6. (a) $x = t$ and $y = t^2$, where $t \in \mathbb{R}$ (b) $x = t^2$ and $y = t$, where $t \in \mathbb{R}$
(c) $x = t^2$ and $y = t$, where $t \geq 0$ (d) $x = t^2$ and $y = t$, where $t \leq 0$
(e) $x = t$ and $y = 2^t$, where $t \in \mathbb{R}$ (f) $x = t$ and $y = \log t$, where $t > 0$

7. (a) $x = y^2$, which is a parabola with axis of symmetry being the x-axis, opening to the right; a particle moves to the left along the lower branch of the parabola, reaches the origin at $t = 0$, and then moves to the right along the upper branch of the parabola. The speed of the particle is not constant, so we'll leave a determination of a formula for its speed to a future calculus course.

(b) $x^2 + y^2 = 1$, which is a circle of radius 1 unit, with centre at the origin; a particle moves along the circle counter-clockwise, making one complete revolution in 2π time units, which means the particle's speed is 1.

(c) $x^2 + y^2 = 1$, which is a circle of radius 1 unit, with centre at the origin; a particle moves along the circle counter-clockwise, making one complete revolution in π time units, which means the particle's speed is 2.

(d) $x^2 + y^2 = 1$, which is a circle of radius 1 unit, with centre at the origin; a particle moves along the circle clockwise, making one complete revolution in 2π time units, which means the particle's speed is 1.

(e) $x^2 + y^2 = 9$, which is a circle of radius 3 units, with centre at the origin; a particle moves along the circle counter-clockwise, making one complete revolution in 1 time units, which means the particle's speed is 6π.

(f) $x^2 + y^2 = r^2$, which is a circle of radius r units, with centre at the origin; a particle moves along the circle counter-clockwise if $f > 0$ and clockwise if $f < 0$, making one complete revolution in $\dfrac{1}{|f|}$ time units, which means the particle's speed is $2\pi r|f|$.

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