Linear Algebra, Part 1

Tutorial 2-3: Standard Equations of Lines

In this tutorial you will review how to calculate the slope of a line, and various forms of equations of lines. Answers to the exercises are found at the end of the document.

1. The slope of a line is calculated using the "rise-over-run" idea, as illustrated in Figure 1. Use this idea to calculate the slope in each case.

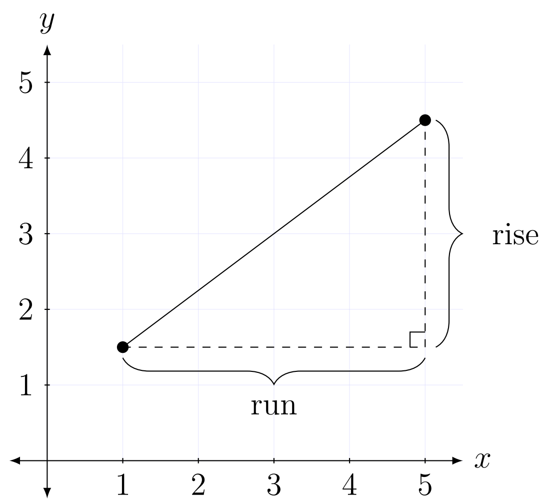


Figure 1: The diagram illustrates the calculation of the slope of the line that passes through the points $(1, 2)$ and $(5, 4)$. The slope of this line is $\dfrac{\text{rise}}{\text{run}} = \dfrac{3}{4}$.

   (a) Calculate the slope of the line that includes the points $(1, 2)$ and $(4, 7)$.

   (b) Calculate the slope of the line that includes the points $(-1, 3)$ and $(1, 4)$.

   (c) Calculate the slope of the line that includes the points $(-2, 4)$ and $(3, -5)$.

   (d) Calculate the slope of the line that includes the points $(3, 2)$ and $(5, 2)$.

   (e) Calculate the slope of the line that includes the points $(2, -1)$ and $(2, 4)$.

   (f) Write a formula for the slope of the line that includes the points (x_1, y_1) and (x_2, y_2).

   (g) State the slope of the line that has formula $(y = 2x - 3)$.

   (h) Calculate the slope of the line that has formula $(3x + 4y = 5)$.

2. As you will recall from previous studies, functions of the form $y = mx + b$ have graphs that are straight lines. You might also recall the phrase from geometry that "two points determine a line;" in other words, once you identify two points on a line, there is one and only one line that passes through the two points. (You can place your ruler on the two points and sketch this unique straight line.) Given a formula, among the easiest two points to determine can be found by substituting first $x = 0$ into the formula and then solving for y, and then substituting $y = 0$ into the formula and solving for x. This will determine the y-intercept and the x-intercept.

   Sketch each line.

   (a) $y = 2x - 3$

   (b) $y = x + 2$

   (c) $y = -x - 1$

   (d) $y = -2x + 3$

3. In the previous exercise, determine the slope and the y-intercept for each function.

4. (a) For a function of the form $y = mx + b$, state the graphical meanings of the parameters m and b, based on your calculations in the previous exercise.

   (b) Prove that the meaning of the parameter m in Part (a) is what you stated by considering two points (x_1, y_1) and (x_2, y_2), where $x_2 \neq x_1$, that lie on the line that has formula $y = mx + b$.

5. Because two points determine a line, we should be able to determine an equation for a line that passes through two points if we are given the coördinates of the two points. One way to do this is to first use the coördinates of the two points to determine the slope of the line, and then substitute the value of the slope and the coördinates of either point into the template formula $y = mx + b$ to determine the value of b.

   Determine an equation for the line that passes through each pair of points.

   (a) $(0, -4)$ and $(2, 0)$

   (b) $(1, 0)$ and $(3, 6)$

   (c) $(-1, -2)$ and $(3, -5)$

   (d) $(2, 4)$ and $(-3, -3)$

   (e) $(4, 3)$ and $(9, 3)$

   (f) $(2, 0)$ and $(6, 0)$

6. Note from the previous exercise that horizontal lines have equations of the form $y = b$, and conversely, equations of the form $y = b$ represent horizontal lines for all real values of the parameter b. The method used in the previous exercise fails if the two given points lie along a vertical line; try it yourself for the two points $(2, 0)$ and $(2, 5)$.

   Determine an equation for the line that passes through each pair of points.

(a) $(3, -4)$ and $(3, 0)$

(b) $(2, 5)$ and $(2, 6)$

(c) $(-1, -2)$ and $(-1, -5)$

(d) $(0, 4)$ and $(0, -3)$

7. You will have noted that in Parts (c) and (d) of the previous exercise the equations for the lines do not fit into the pattern of $y = mx + b$. The reason for this is that the lines are vertical and therefore the slope of the line does not exist. Vertical lines have equations of the form $x =$ constant. Parts (a) and (b) of the previous exercise remind us that horizontal lines have equations of the form $y =$ constant. Horizontal lines have slope $m = 0$, and so their equations do fit into the pattern $y = mx + b$, with the special case that $m = 0$. Thus, formulas of the form $y = mx + b$ include all lines except vertical ones.

   A more general "template" form of the equation of a straight line is

   $$Ax + By = C$$

   which includes all possible lines in a plane.

   (a) What are the values of the parameters A, B, and C for a horizontal line?

   (b) What are the values of the parameters A, B, and C for a vertical line?

8. Determine the slope and y-intercept for each line.

   (a) $2x + 3y = 4$

   (b) $2x - 3y = 4$

   (c) $3x + 4y = -2$

   (d) $3x - 4y = 2$

9. Sketch a graph for each line in the previous exercise.

10. Consider a line described by $Ax + By = C$.

   (a) Determine an expression for the slope of the line in terms of the parameters A, B, and C.

   (b) Determine an expression for the y-intercept of the line, in terms of the parameters A, B, and C.

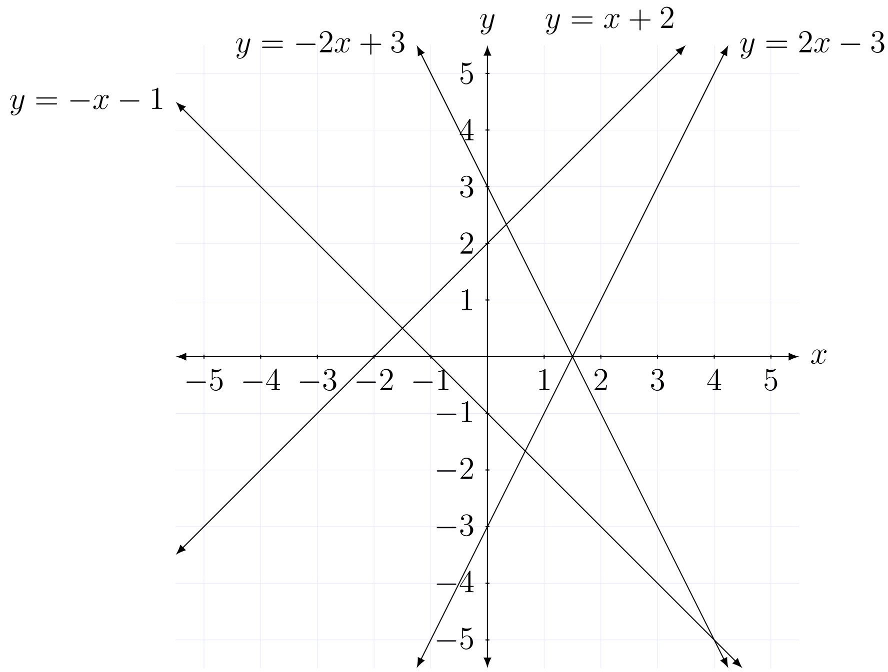
   (c) Determine an expression for the x-intercept of the line, in terms of the parameters A, B, and C.

Answers to exercises:

1. (a) $\frac{5}{3}$ (b) $\frac{1}{2}$ (c) $-\frac{9}{5}$ (d) 0 (e) the slope does not exist

(f) $\dfrac{y_2 - y_1}{x_2 - x_1}$ (g) 2 (h) $-\dfrac{3}{4}$

2.



3. (a) slope = 2, y-intercept = −3 (b) slope = 1, y-intercept = 2
(c) slope = −1, y-intercept = −1 (d) slope = −2, y-intercept = 3

4. (a) The value of m is the slope of the graph, and the value of b is the y-intercept of the graph.

(b) The solution to this exercise is a proof that the value of the parameter m in the formula $y = mx + b$ is the slope of the corresponding graph.

Because the point (x_1, y_1) lies on the graph of the line that has formula $y = mx + b$, it follows that

$$y_1 = mx_1 + b$$

Because the point (x_2, y_2) also lies on the graph of the line that has formula $y = mx + b$, it follows that

$$y_2 = mx_2 + b$$

The slope of the line that passes through the points (x_1, y_1) and (x_2, y_2) is defined to be

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Therefore,

$$\begin{aligned}
\frac{y_2 - y_1}{x_2 - x_1} &= \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} \\
\frac{y_2 - y_1}{x_2 - x_1} &= \frac{mx_2 + b - mx_1 - b}{x_2 - x_1} \\
\frac{y_2 - y_1}{x_2 - x_1} &= \frac{mx_2 - mx_1}{x_2 - x_1} \\
\frac{y_2 - y_1}{x_2 - x_1} &= m\left(\frac{x_2 - x_1}{x_2 - x_1}\right) \\
\frac{y_2 - y_1}{x_2 - x_1} &= m
\end{aligned}$$

Note that the proof only makes sense provided that $x_2 \neq x_1$.

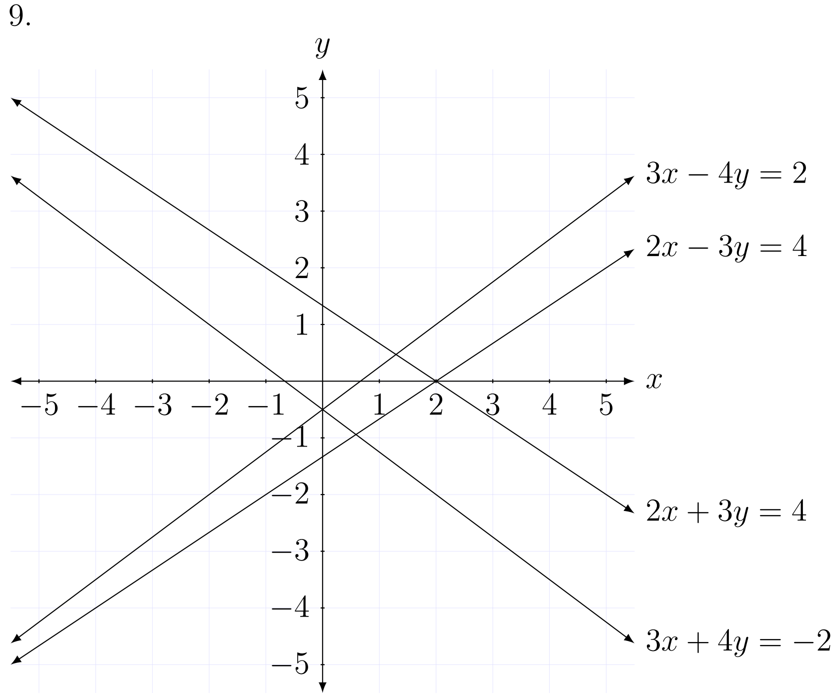
5. (a) $y = 2x - 4$ (b) $y = 3x - 3$ (c) $y = -\frac{3}{4}x - \frac{11}{4}$
(d) $y = \frac{7}{5}x + \frac{6}{5}$ (e) $y = 3$ (f) $y = 0$

6. (a) $x = 3$ (b) $x = 2$ (c) $x = -1$ (d) $x = 0$

7. (a) $A = 0$, $B \neq 0$, and C can have any value.
(b) $A \neq 0$, $B = 0$, and C can have any value.

8. (a) slope $= -\frac{2}{3}$, y-intercept $= \frac{4}{3}$ (b) slope $= \frac{2}{3}$, y-intercept $= -\frac{4}{3}$
(c) slope $= -\frac{3}{4}$, y-intercept $= -\frac{1}{2}$ (d) slope $= \frac{3}{4}$, y-intercept $= -\frac{1}{2}$

9.



10. (a) The line will have a slope provided that $B \neq 0$. In this case,

$$
\begin{aligned}
Ax + By &= C \\
By &= -Ax + C \\
y &= -\frac{A}{B}x + \frac{C}{B}
\end{aligned}
$$

Thus, provided that $B \neq 0$, the slope of the line is $-\dfrac{A}{B}$.

(b) Based on the result of Part (a), provided that $B \neq 0$, the y-intercept of the line is $\dfrac{C}{B}$.

(c) Setting $y = 0$ in $Ax + By = C$ and solving for x, we can see that (provided that $A \neq 0$), the x-intercept is $\dfrac{C}{A}$.

For the latest version of these notes, see:

`https://fomap.org/prepare-for-university/linear-algebra-part-1/`