Linear Algebra, Part 1

Tutorial 2-2: Analytic Geometry

In this tutorial you will review some basic facts from analytic geometry, particularly about lines, that are relevant for our study of linear algebra. Answers to the exercises are found at the end of the document.

The first works of geometry, which date back to the times of the ancient Greek thinkers, more than 2000 years ago, used logic and reason to prove that geometric figures have various properties. (We now call this method in geometry, without using coördinates or formulas, synthetic geometry.) The ancient Greek mathematicians Menaechmus and Apollonius did preliminary work to connect geometric figures with numerical calculation, and further work in this direction was done by Omar Khayyam in the 1000s CE. Decisive steps were taken independently by Fermat and Descartes in the 1600s CE, in work that was systematized by Euler in the following century. By this time using coördinates to identify points in space (what we now call Cartesian coördinates in honour of Descartes) was established, and the use of letters to represent unknown quantities was also well-established, paving the way to use algebraic techniques to solve geometry problems. This use of coördinates and algebra to solve geometry problems is called analytic geometry.

1. **Pythagoras's theorem** Using Pythagoras's theorem,

   $$c^2 = a^2 + b^2$$

   one can calculate the length of one side of a right triangle given the lengths of the other two sides.

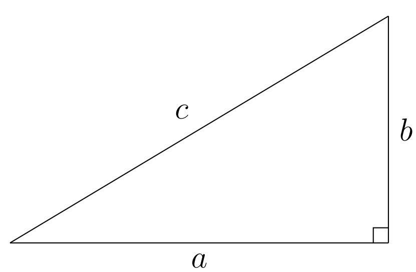

Figure 1: A triangle is a right triangle if and only if the sides are related by Pythagoras's theorem: $c^2 = a^2 + b^2$.

   (a) Determine the length of the hypotenuse of a right triangle if the lengths of the two other sides of the triangle are 1.80 cm and 2.50 cm.

(b) Determine the length of one side of a right triangle if the length of the hypotenuse is 7.80 m and the length of the other side is 3.40 m.

2. **Distance between two points (length of a line segment)** To determine the distance between two points, if the coördinates of the two points are known, one can proceed by constructing a suitable right triangle with the two points as the endpoints of the hypotenuse of the triangle, and then applying Pythagoras's theorem. By "suitable" one means "in a way that is as easy as possible," which in this case means to arrange for one of the shorter sides of the right triangle to be parallel to the x-axis and the other of the shorter sides of the right triangle to be parallel to the y-axis. See Figure 2.

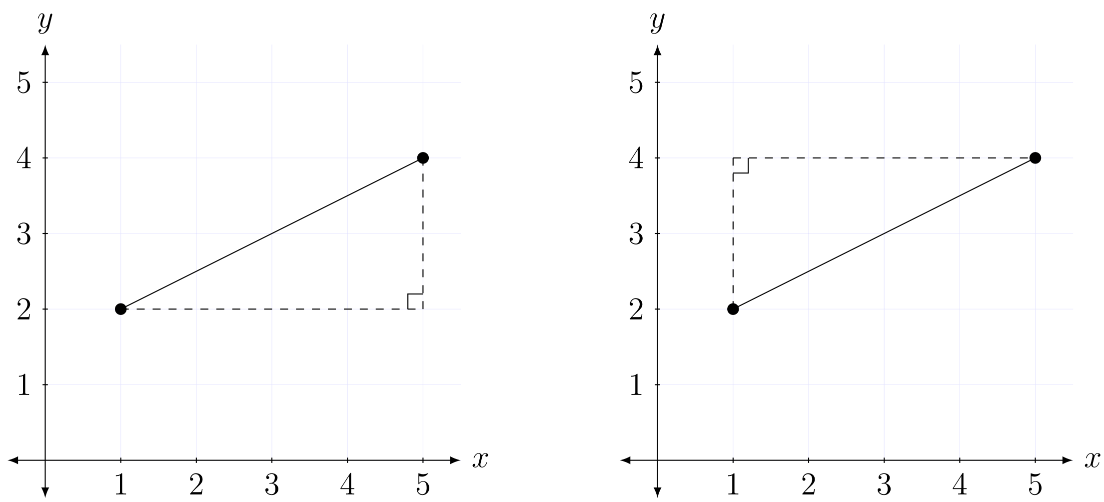

Figure 2: These diagrams illustrate the easiest ways to apply Pythagoras's theorem to calculate the distance between the indicated points, which have coördinates $(1, 2)$ and $(5, 4)$. Construct a right triangle using sides that are parallel to the coördinate axes, according to either the diagram on the left or the diagram on the right, whichever you prefer. The distance between the indicated points is $\sqrt{4^2 + 2^2} = \sqrt{20}$.

Apply the idea illustrated in Figure 2 to calculate each of the indicated lengths. Sketching a diagram is recommended in each case!

(a) Calculate the distance between the point $(1, 5)$ and the point $(3, 2)$.

(b) Calculate the distance between the point $(-1, 6)$ and the point $(3, -2)$.

(c) Calculate the distance between the point $(-2, -4)$ and the point $(5, 6)$.

(d) Calculate the distance between the point $(1, 3)$ and the point $(1, 8)$.

(e) Calculate the distance between the point $(1, 4)$ and the point $(7, 4)$.

(f) Write a formula for the distance between the point (x_1, y_1) and the point (x_2, y_2).

3. **Midpoint of a line segment** Consider the line segment AB indicated by the solid line in Figure 3. Imagine walking a path that starts at one endpoint of the line segment, say A, and ends at the other end of the line segment, B, but that proceeds along the dashed line segments. That is, the first leg of the journey is to walk along the horizontal dashed line from A to D, and then the second leg of the journey is to walk along the vertical dashed line from D to C. Is it intuitively clear to you that to walk from A to the midpoint of the line segment AB you can first walk half-way along the horizontal black dashed line (i.e., to D) and then turn and walk along the vertical red dashed line until you reach the solid line segment? You can write this conclusion in general using a nice formula as follows: The midpoint of the line segment that has endpoints (x_1, y_1) and (x_2, y_2) has coördinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

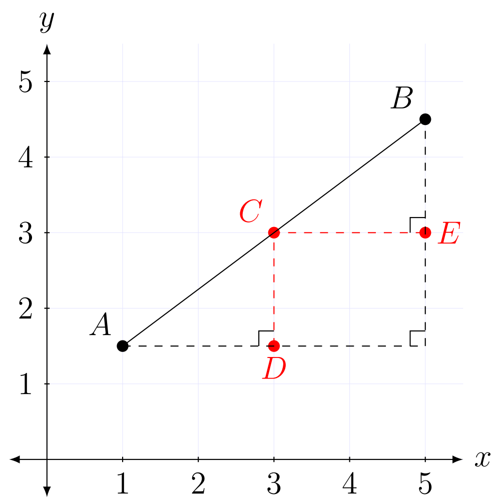

Figure 3: The midpoint of the line segment that has endpoints $(1, 2)$ and $(5, 4)$ has coördinates $(3, 3)$. The x-coördinate of the midpoint is the mean of the x-coördinates of the endpoints, and the y-coördinate of the midpoint is the mean of the y-coördinates of the endpoints.

Whether this is intuitively clear or not, you can prove that this is so in at least two ways. One way is to apply a little bit of Euclidean geometry to prove that the two smaller right triangles in the figure (i.e., ΔADC and ΔCEB) are congruent, from which it follows that the point C is indeed the midpoint of line segment AB. A second way is to use the distance formula and algebra to show that the distance from the point $A(x_1, y_1)$ to the point $C\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ is equal to the distance from the point $C\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ to the point $B(x_2, y_2)$. (If you are a stickler for detail then you will also desire to prove that the point C really does lie on the line segment AB.)

Use these ideas to determine the coördinates of the midpoints of the line segments in each case.

(a) Determine the coördinates of the midpoint of the line segment that has endpoints $(1, 2)$ and $(1, 6)$.

(b) Determine the coördinates of the midpoint of the line segment that has endpoints $(1, 2)$ and $(8, 2)$.

(c) Determine the coördinates of the midpoint of the line segment that has endpoints $(3, 2)$ and $(6, 1)$.

(d) Determine the coördinates of the midpoint of the line segment that has endpoints $(-1, 5)$ and $(-4, -2)$.

4. The same idea as in the previous exercise can be used to determine the coördinates of a point that is a fraction of the distance from one given point to another given point. For example, you can show that the coördinates of the point C that is one-nth the distance from point $A(x_1, y_1)$ to point $B(x_2, y_2)$ along the line segment AB are

$$\left(x_1 + \frac{x_2 - x_1}{n}, y_1 + \frac{y_2 - y_1}{n}\right) \quad \text{which is equivalent to} \quad \left(\frac{(n-1)x_1 + x_2}{n}, \frac{(n-1)y_1 + y_2}{n}\right)$$

(a) Determine the coördinates of the point on the line segment that has endpoints $A(1, 3)$ and $B(2, 8)$ that is one-third of the distance from A to B.

(b) Determine the coördinates of the point on the line segment that has endpoints $A(2, -3)$ and $B(4, 6)$ that is one-quarter of the distance from A to B.

(c) Determine the coördinates of the point on the line segment that has endpoints $A(10, 2)$ and $B(4, -1)$ that is two-thirds of the distance from A to B.

(d) Determine the coördinates of the point on the line segment that has endpoints $A(-5, 3)$ and $B(8, 6)$ that is 82% of the distance from A to B.

5. Show that the point with coördinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ lies on the line segment with endpoints (x_1, y_1) and (x_2, y_2).

6. Use the distance formula and algebra to show that the point with coördinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ is the midpoint of the line segment that has endpoints (x_1, y_1) and (x_2, y_2).

7. Show that the point with coördinates $\left(x_1 + \frac{x_2 - x_1}{n}, y_1 + \frac{y_2 - y_1}{n}\right)$ lies on the line segment with endpoints (x_1, y_1) and (x_2, y_2).

8. Consider the points $A(x_1, y_1)$, $B(x_2, y_2)$, and $C\left(x_1 + \frac{x_2 - x_1}{n}, y_1 + \frac{y_2 - y_1}{n}\right)$. Use the distance formula and algebra to show that

$$\text{length of } AC = \frac{1}{n} \times \text{length of } AB$$

Answers to exercises:

1. (a) 3.08 cm (b) 7.02 m
2. (a) $\sqrt{13}$ (b) $4\sqrt{5}$ (c) $\sqrt{149}$ (d) 5 (e) 6 (f) $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
3. (a) $(1, 4)$ (b) $(4.5, 2)$ (c) $(4.5, 1.5)$ (d) $(-2.5, 1.5)$
4. (a) $\left(\frac{4}{3}, \frac{14}{3}\right)$ (b) $\left(\frac{5}{2}, -\frac{3}{4}\right)$ (c) $(6, 0)$ (d) $(5.66, 5.46)$

5. First determine an equation for the line that passes through the points (x_1, y_1) and (x_2, y_2). You can do this in steps; in the first step determine an expression for the slope m of the line:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

In the next step, consider an arbitrary point on the line, (x, y), and note that because this point lies on the line we can equally well write the following expression for the slope of the line:

$$m = \frac{y - y_1}{x - x_1}$$

Equating the two expressions for the slope of the line produces an equation for the line:

$$\begin{aligned}
\frac{y - y_1}{x - x_1} &= \frac{y_2 - y_1}{x_2 - x_1} \\
y - y_1 &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) \\
y &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) + y_1
\end{aligned}$$

The next step is to determine the y-value for the point on the line that has x-value

$$x = \frac{x_1 + x_2}{2}$$

Substituting this expression for x into the equation for the line, we obtain

$$\begin{aligned}
y &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)\left(\frac{x_1 + x_2}{2} - x_1\right) + y_1 \\
y &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)\left(\frac{x_1 + x_2}{2} - \frac{2x_1}{2}\right) + y_1 \\
y &= \left(\frac{y_2 - y_1}{2}\right) + y_1 \\
y &= \left(\frac{y_2 - y_1}{2}\right) + \frac{y_1}{2} \\
y &= \frac{y_1 + y_2}{2}
\end{aligned}$$

This proves that the point with coördinates $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$ lies on the line segment with endpoints (x_1, y_1) and (x_2, y_2).

6. The length L of the line segment that has endpoints (x_1, y_1) and (x_2, y_2) is the distance between these two points:

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance M between one endpoint (x_1, y_1) and the point $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$ is

$$\begin{aligned}
M &= \sqrt{\left(\frac{x_1+x_2}{2} - x_1\right)^2 + \left(\frac{y_1+y_2}{2} - y_1\right)^2} \\
M &= \sqrt{\left(\frac{x_1+x_2}{2} - \frac{2x_1}{2}\right)^2 + \left(\frac{y_1+y_2}{2} - \frac{2y_1}{2}\right)^2} \\
M &= \sqrt{\left(\frac{x_2-x_1}{2}\right)^2 + \left(\frac{y_2-y_1}{2}\right)^2} \\
M &= \sqrt{\frac{(x_2-x_1)^2}{4} + \frac{(y_2-y_1)^2}{2}} \\
M &= \frac{1}{2}\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \\
M &= \frac{1}{2}L
\end{aligned}$$

This completes the proof that the point with coördinates $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$ is the midpoint of the line segment that has endpoints (x_1, y_1) and (x_2, y_2).

7. This proof uses the same procedure as in Exercise 6. We have already (in the solution to Exercise 6) determined that an equation for the line that passes through the points (x_1, y_1) and (x_2, y_2) is

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) + y_1$$

Next we'll determine the y-value for the point on the line that has x-value

$$x = x_1 + \frac{x_2 - x_1}{n}$$

by substituting this expression for x in the equation of the line and then simplifying:

$$
\begin{aligned}
y &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)\left(x_1 + \frac{x_2 - x_1}{n} - x_1\right) + y_1 \\
y &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)\left(\frac{x_2 - x_1}{n}\right) + y_1 \\
y &= \frac{y_2 - y_1}{n} + y_1 \\
y &= y_1 + \frac{y_2 - y_1}{n}
\end{aligned}
$$

This completes the proof that the point with coördinates $\left(x_1 + \frac{x_2 - x_1}{n}, y_1 + \frac{y_2 - y_1}{n}\right)$ lies on the line segment with endpoints (x_1, y_1) and (x_2, y_2).

8. This proof uses the same procedure as in Exercise 7. The length of the line segment that has endpoints (x_1, y_1) and (x_2, y_2) is the distance between these two points:

$$
\text{length of } AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
$$

The length of the line segment AC is the distance between the point $A(x_1, y_1)$ and the point $C\left(x_1 + \frac{x_2 - x_1}{n}, y_1 + \frac{y_2 - y_1}{n}\right)$, which is

$$
\begin{aligned}
\text{length of } AC &= \sqrt{\left(x_1 + \frac{x_2 - x_1}{n} - x_1\right)^2 + \left(y_1 + \frac{y_2 - y_1}{n} - y_1\right)^2} \\
\text{length of } AC &= \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2} \\
\text{length of } AC &= \sqrt{\frac{(x_2 - x_1)^2}{n^2} + \frac{(y_2 - y_1)^2}{n^2}} \\
\text{length of } AC &= \frac{1}{n}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
\text{length of } AC &= \frac{1}{n} \times \text{length of } AB
\end{aligned}
$$

This completes the proof.

For the latest version of these notes, see:

`https://fomap.org/prepare-for-university/linear-algebra-part-1/`