Linear Algebra, Part 1

Tutorial 2-1: Cartesian Coördinates

In this tutorial you will review Cartesian coördinates in a plane, and then explore using Cartesian coördinates in three dimensions. This will be valuable for our studies of linear algebra in the next chapter. Answers to the exercises are found at the end of the document.

What we nowadays call Cartesian (in honour of Descartes) coördinate systems are a pair of straight lines in a plane that intersect at right angles (or three straight lines that intersect in a single point and are mutually perpendicular in three-dimensional space), each having a measurement scale, like the number lines we recall from elementary school.

Remember that every point has unique coördinates, and every set of coördinates corresponds to a unique point, in the same way that every home has a unique address, and every address corresponds to a unique home.

We'll follow the usual convention of drawing one coördinate axis horizontal, and labelling it the x-axis, and the other coördinate axis vertical, and labelling it the y-axis. The point of intersection of the x-axis and the y-axis is called the *origin* of the coördinate system, and has coördinates $(0, 0)$.

1. State the coördinates of the points plotted in Figure 1.

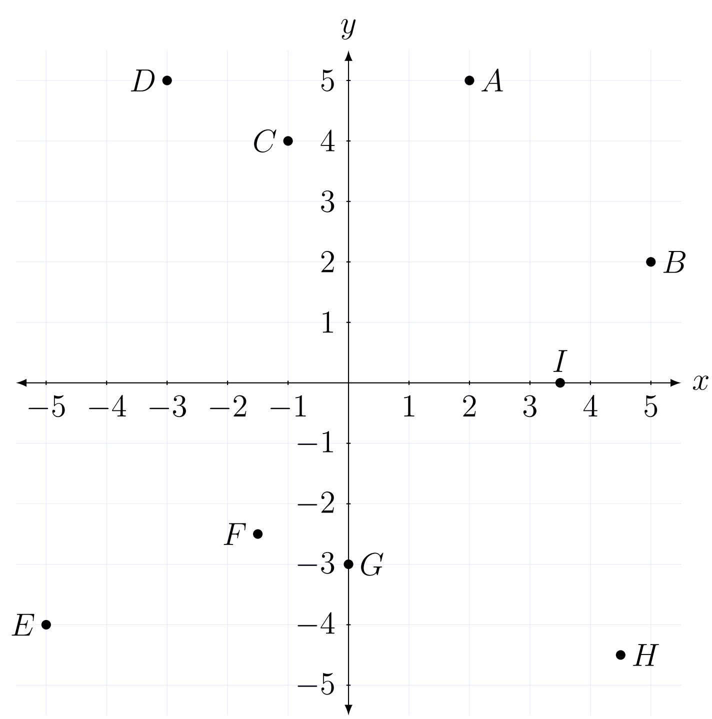


Figure 1: Points plotted in a two-dimensional plane that has a Cartesian coördinate system.

2. Plot each point in a plane, using a Cartesian coördinate system.
   (a) $A(4, 1)$
   (b) $B(-3, 2)$
   (c) $C(-2, -4)$
   (d) $D(3, -2)$
   (e) $E(0, 0)$
   (f) $F(0, 2.5)$
   (g) $G(-2.5, 0)$

3. Describe the collection of points in a plane that satisfy each condition. Then plot each collection of points.
   (a) $x = 0$
   (b) $x = 2$
   (c) $x = 4$
   (d) $x = -1.5$
   (e) $y = 0$
   (f) $y = 1$
   (g) $y = 3$
   (h) $y = -2.5$

4. The xy-coördinates that we have reviewed are suitable for specifying the locations of points in a plane. We live in a three-dimensional space, and so to describe points in space we need three coördinates, which are typically called x, y, and z.

   A good way to visualize the three mutually perpendicular Cartesian coördinate axes in three-dimensional space is to place a sheet of paper on your desk and sketch perpendicular x-axis and y-axis on it. As usual, the positive side of the x-axis is to the right of the origin, and the negative side is to the left of the origin. The positive side of the y-axis is on the side of the origin towards the top of the page, and the negative side is on the side of the origin towards the bottom of the page. The z-axis passes through the origin and is perpendicular to your desk. You can imagine this line passing through the origin extending above the desk (this is the positive side) and also below the desk (this is the negative side). You might actually place a stick, or pencil, or ruler, standing vertically on your desk with its base at the origin of the coördinate system, to represent the positive z-axis.

   An alternative is to sketch a perspective drawing, such as the one in Figure 2. This is popular in textbooks, and is the best one can do on flat pages, but I find visualizing using my desk more effective. Nevertheless, you should get used to identifying coördinates in figures like Figure 2, because the university textbooks you will read (and the software used in university courses) will have figures using this perspective.

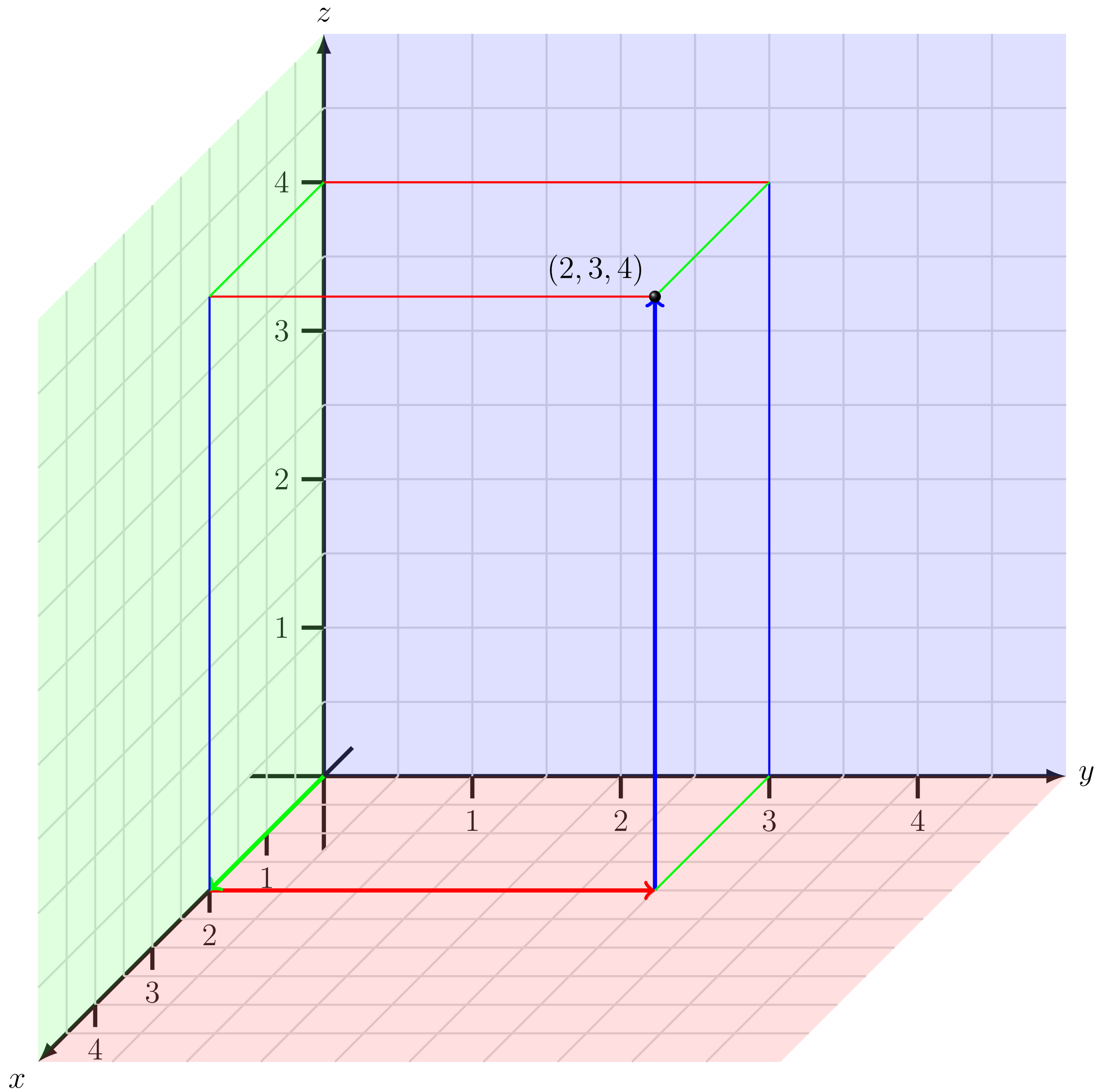


Figure 2: The figure shows a perspective drawing of a three-dimensional Cartesian coördinate system. This differs from the perspective that one typically uses with a sheet of paper on one's desk for the xy-plane, because the sheet of paper has been rotated 90° clockwise, so that you see in the foreground the octant for which all coördinates are positive.

Note the coloured guidelines in Figure 2, that outline a rectangle. You might sketch similar rectangles in your own perspective drawings to help you visualize the locations of various points.

In a plane, the two coördinate axes separate the plane into four quadrants. In three-dimensional space, the xy-plane, the xz-plane, and the yz-plane separate space into eight octants.

Plot each point in three-dimensional space, using a Cartesian coördinate system. Alternatively, use a sheet of paper on your desk to represent the xy-plane and describe the locations of the points in space relative to the origin on your desk.

(a) $A(2, 2, 2)$
(b) $B(2, 2, -2)$
(c) $C(2, -2, 2)$
(d) $D(2, -2, -2)$
(e) $E(-2, 2, 2)$
(f) $F(-2, 2, -2)$
(g) $G(-2, -2, 2)$
(h) $H(-2, -2, -2)$

5. Describe the collection of points in three-dimensional space that satisfy each condition.

   (a) $x = 0$
   (b) $x = 2$
   (c) $x = -3$
   (d) $y = 0$
   (e) $y = 1$
   (f) $y = -2$
   (g) $z = 0$
   (h) $z = 3$
   (i) $z = -1$
   (j) $y = x$
   (k) $y = z$
   (l) $x = z$

6. Practice looking at the world in terms of three-dimensional coördinate systems. Whenever you are at a desk or table, practice choosing an origin on the surface of the desk and identifying the signs of the three coördinates at various positions above and below the desk. You can also do this when you look at rooms by choosing an origin somewhere in the room. What if you choose a corner of the room to be the origin? Does this make your task more interesting, easier, or harder? Play with this!

7. You might also look for the presence of coördinates or something like them in everyday life. For example, some cities have streets that are (at least partially) laid out in a rectangular grid, and are similarly named. A famous example is Manhattan, but an even nicer example is Calgary, in which the city is separated into four quadrants, named NW, NE, SW, and SE, much like the four quadrants in a Cartesian coördinate system. If you search a map of either of these cities you'll have fun.

   If you have played the board game "Battleships" then you have another example of using coördinates to communicate locations of objects located on a grid.

Archaeologists use grids to organize the excavation of a site, and to label artifacts that are removed from the site, so that later the original relative locations of the artifacts can be reconstructed.

Chess is dear to me, and since about 1980 using a grid system for reporting the moves played in a chess game has become standard. The current standard has become called algebraic notation, but it is interesting to note that the origins of using grid systems for chess notation date at least as far back as 1520; consult the article linked earlier in this paragraph for further details.

Coördinates are useful and precise, and they allow us to apply numerical and algebraic tools to geometry, so it's no surprise that they are essential in mathematics, but their usefulness extends to many other fields.

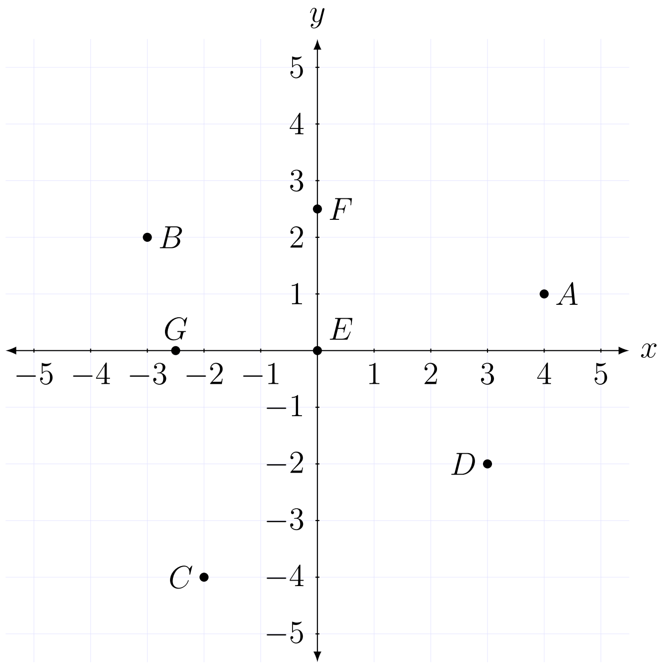
8. Can you visualize four-dimensional space? I can't, and I don't know anyone who can. (Maybe you can? This might be worth thinking about a little bit, but it may not be possible for us limited humans ... .) There are various tricks (akin to perspective diagrams on a sheet of paper that represent three-dimensional geometrical figures, or artistic techniques to give the illusion of three dimensions in a flat drawing) that might help one get a sense for four dimensions, but they fall short of actual visualization. Nevertheless, spaces of four dimensions or more are useful in both mathematics and science for various purposes, and even if we can't visualize them, we can learn to work with them using numerical or algebraic techniques. It's good to keep this in mind as we pursue our study of linear algebra, where we will begin learning how to deal with spaces of higher dimensions.

Answers to exercises:

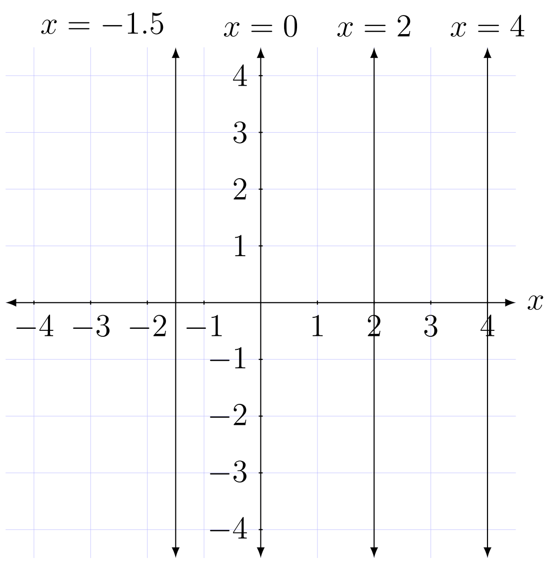
1. $A(2,5)$; $B(5,2)$; $C(-1,4)$; $D(-3,5)$; $E(-5,-4)$

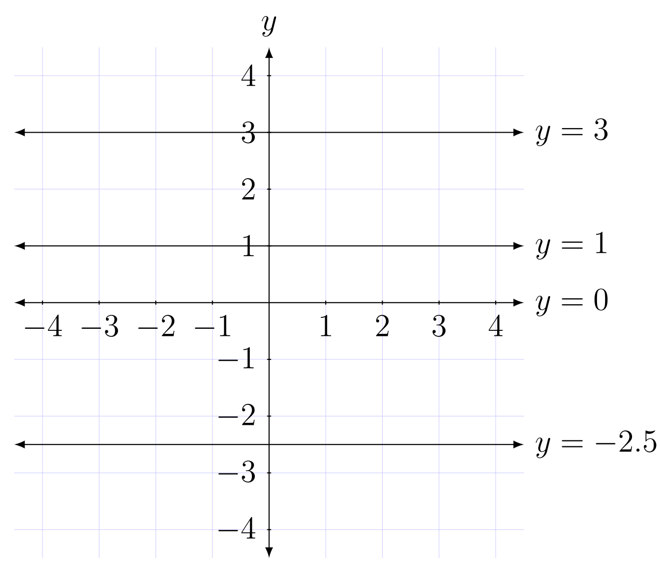
$F(-1.5,-2.5)$; $G(0,-3)$; $H(4.5,-4.5)$; $I(3.5,0)$

2.



3. (a) The y-axis.
(b) The line parallel to the y-axis and 2 units "east" of the y-axis.
(c) The line parallel to the y-axis and 4 units "east" of the y-axis.
(d) The line parallel to the y-axis and 1.5 units "west" of the y-axis.





(e) The x-axis.
(f) The line parallel to the x-axis and 1 units "north" of the x-axis.
(g) The line parallel to the x-axis and 3 units "north" of the x-axis.
(h) The line parallel to the x-axis and 2.5 units "south" of the x-axis.

4. Imagine that the top of your desk is the xy-plane, and has a Cartesian coördinate system sketched on it. In the following sentences "above" means above your desk and "below

means below your desk.

(a) Two units above the point $(2, 2)$.
(b) Two units below the point $(2, 2)$.
(c) Two units above the point $(2, -2)$.
(d) Two units below the point $(2, -2)$.
(e) Two units above the point $(-2, 2)$.
(f) Two units below the point $(-2, 2)$.
(g) Two units above the point $(-2, -2)$.
(h) Two units below the point $(-2, -2)$.

5. Imagine that the top of your desk is the xy-plane, and has a Cartesian coördinate system sketched on it.

(a) This is the yz- plane. You can model this by using a sheet of paper or a book that has one edge on your desk, includes both the y-axis and the z-axis, and extends indefinitely to fill the entire yz-plane.

(b) You can model this using a sheet of paper that is parallel to the yz-plane, but 2 units east of the yz-plane.

(c) You can model this using a sheet of paper that is parallel to the yz-plane, but 3 units west of the yz-plane.

(d) This is the xz- plane. You can model this by using a sheet of paper or a book that has one edge on your desk, includes both the x-axis and the z-axis, and extends indefinitely to fill the entire xz-plane.

(e) You can model this using a sheet of paper that is parallel to the xz-plane, but 1 unit north of the xz-plane.

(f) You can model this using a sheet of paper that is parallel to the xz-plane, but 2 units south of the xz-plane.

(g) This is the xy- plane. You can model this as the top of your desk, but it extends indefinitely to fill the entire xz-plane.

(h) You can model this using a sheet of paper that is parallel to the xy-plane, but 3 units vertically above the xy-plane.

(i) You can model this using a sheet of paper that is parallel to the xy-plane, but 1 unit vertically below the xy-plane.

(j) You can model this using a sheet of paper that is "standing up" vertically on your desk, with one edge along the line $y = x$, extending indefinitely above and below your desk.

(k) You can model this using a sheet of paper that has one edge along the x-axis, and then is tipped "forward" by 45°, so that an edge of the sheet (let's say the left edge, although it could be the right edge, or any other line on the paper that is parallel to the left and right edges) lies along the line $y = z$. As usual, you must use your imagination to extend the sheet of paper indefinitely to fill the entire plane.

(l) You can model this using a sheet of paper that has one edge along the y-axis, and then is tipped "to the right" by 45°, so that an edge of the sheet (let's say the front edge, although it could be the back edge, or any other line on the paper that is parallel to the front and back edges) lies along the line $x = z$. As usual, imagine the sheet of paper extending indefinitely to fill the entire plane.

6. Play with this!

7. Explore this!
8. Play with this!

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