Patterns in the Multiplication Table

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Learning mathematics is much like learning to play a musical instrument, learning one of the fine arts, or learning to play a sport or practicing a martial art. There is a certain amount of technical material (such as scales in music) that one must practice over and over in order to internalize the techniques and make them automatic. Additionally, there is conceptual understanding in mathematics, consisting of problem solving, mathematical modeling, etc., which is analogous to musicality or learning to play songs or classical pieces.

In the past thirty years or so, there has been quite a war in the mathematics education community over which of these two aspects is more important, and which ought to be emphasized in teaching mathematics. Some educators believe in "back to basics" and other believe that technical skills are not important, and are better left to calculators and computers, so that students can focus on higher-level thinking skills.

This is a false dichotomy; both technical skills and higher-level thinking skills are important, and they reinforce each other. Someone with excellent technical skills has an advantage in developing higher-level thinking skills, whereas exercising higher-level thinking skills motivates the need for technical ability. And certainly an unending stream of technical exercises is bound to kill nearly everyone's enthusiasm for learning.

One way of approaching this false dichotomy is by asking the question, "How much should a mathematics student memorize?" A back-to-basics person might answer, "A lot," and the opposite type might answer, "very little." Let's consider the multiplication table as an example. Should children memorize it by rote? A back-to-basics person would say yes, an opposite type would say that it is not important to know the multiplication table, as a student can use a calculator, and the time would be better spent on higher-level thinking.

I believe that having an instant recall of the multiplication table is very helpful in learning higher mathematics. But I also believe that there is a better way to learn the multiplication table than rote repetition. If a student can learn to play with the multiplication table, to make friends with it, to really understand it, then I believe she or he will naturally remember it, without special effort. But far beyond this, by understanding the table, a student will be optimally prepared for learning mathematics at higher levels, because the patterns in the multiplication table reappear in high-school algebra.

A deep understanding for *why* algorithms (such as those for multiplication) work *is* higher-level thinking. There is a big difference between learning the multiplication table by rote repetition and a deep understanding of the patterns in the table that will lead to a more natural remembering of the table as a meaningful collection of inter-related facts. Rote, repetitive memorization is a difficult way to learn.

An approach to learning mathematics that embraces the need for mastery of basic technical skills and knowledge, but eschews rote memorization, can be summarized by the following slogan:

The more mathematics you understand, the less you need to memorize.

So let's look at some patterns in the multiplication table, with an eye towards understanding the table, and with the hope that understanding the patterns will facilitate remembering the basic multiplication facts, and pave the way for understanding some high-school algebra facts.

First, here is the table, at least as far as 10×10, which is sufficient for all practical purposes:

$\times$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Now let's examine some patterns in the table.

Symmetry in the Table, and the Commutative Property of Multiplication

Notice that the numbers in the table are symmetric with respect to the main diagonal. This means that we could restrict attention to the main diagonal (the blue numbers) and the upper part of the table, if we wish, as in the table on the left, or we could restrict attention to the main diagonal and the lower part of the table, as in the table on the right:

$\times$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2		4	6	8	10	12	14	16	18	20
3			9	12	15	18	21	24	27	30
4				16	20	24	28	32	36	40
5					25	30	35	40	45	50
6						36	42	48	54	60
7							49	56	63	70
8								64	72	80
9									81	90
10										100

$\times$	1	2	3	4	5	6	7	8	9	10
1	1									
2	2	4								
3	3	6	9							
4	4	8	12	16						
5	5	10	15	20	25					
6	6	12	18	24	30	36				
7	7	14	21	28	35	42	49			
8	8	16	24	32	40	48	56	64		
9	9	18	27	36	45	54	63	72	81	
10	10	20	30	40	50	60	70	80	90	100

Both parts of the table are complete in themselves, and it is often helpful to restrict attention to fewer data. Feel free to do this. Mentioning this fact to a student may relieve some anxiety: "Great! Only half as much stuff to worry about than I originally thought!"

And it also illustrates an important property of whole numbers, which is shared by many other number systems, the commutative property of multiplication. Three rows with two dots in each row contains the same number of dots as two rows with three dots in each row; that is, $3 \times 2 = 2 \times 3$. The same is true no matter which whole numbers we use, which explains why we can ignore part of the multiplication table. The number 35 appears twice in the table, once as the result of 5×7, and once as the result of 7×5. Once we know the two results are the same, what is the point of distracting ourselves with redundant information?

Another way to say this is that each number in the table that is NOT on the main diagonal appears twice, just like the number 35. Each number in the table is paired, with one number (such as 18) appearing in the upper part of the table (in the 3×6 position), and its partner number appearing in the lower part of the table (in the 6×3 position for the number 18). But each of the two numbers represents the same multiplication fact, which need not be remembered twice if we understand that the order in which the numbers are multiplied is irrelevant (i.e., the commutative property).

So it would be worthwhile to play with placing coins in rows to illustrate the commutative property of multiplication of whole numbers. For example, place 18 coins in 3 rows with 6 coins in each row, as illustrated in Figure 1.

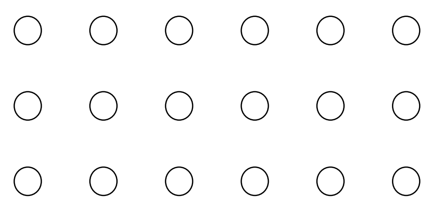

Figure 1: There are 18 coins in total, arranged in 3 rows with 6 coins in each row. But looked at "from the side," the same 18 coins are arranged in 6 rows with 3 coins in each row. This illustrates the commutative law of multiplication of whole numbers.

Now look at the same coins "from the side,"[1] and you will see 6 rows of coins with 3 coins in each row. The number of coins has not changed, only the viewer's perspective, and the commutative law for multiplication of whole numbers has been illustrated for the example $3 \times 6 = 6 \times 3$. Once a student has internalized the commutative property, using arrays of coins or in some other way, you can safely work with your favourite half of the multiplication table, ignoring the other half. (I'll continue to display the full table, as individuals may wish to use either half.)

The visual cortex forms the largest part of the cerebral cortex, and so we ought to make use of visual models to illustrate mathematical principles as much as possible. Even manipulatives have a visual aspect, as they can be seen as well as touched and manipulated. In my experience, using as many diagrams, pictures, and images as possible helps students to understand, internalize, remember, and therefore master mathematical facts and principles.

[1]If you are looking at Figure 1, then just rotate the page by 90° to get the other perspective. If you actually set down the 18 coins, then you can physically move (yourself and your student) to a new vantage point to get the other perspective.

Counting

One of the first mathematical procedures learned by children is to count: $1, 2, 3, 4, \ldots$. Indeed, moving along the first row (from left-to-right), or down the first column of the multiplication table corresponds to "counting by ones." Subsequently, children learn to count by twos, by threes, and so on, and it is useful to remind students that each column and row of the multiplication table can be thought of in terms of something they learned years before.

In the counting-by-twos column or row, the entries are $0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, \ldots$. The last digit of each entry in the list follows the pattern $0, 2, 4, 6, 8, 0, 2, 4, 6, 8, 0, \ldots$. All of the even digits appear in the pattern in the order given.[2]

0 2 4 6 8 0 2 4 6 8 0 2 4 6 8 0 2 4 6 8

Figure 2: The pattern of the last digit of each number in the "counting-by-twos" list.

In the counting-by-fours column or row $(0, 4, 8, 12, 16, 20, 24, 28, 32, 40, \ldots)$, the pattern of the last digits of each entry is $0, 4, 8, 2, 6, 0, 4, 8, 2, 6, 0, \ldots$. The same last digits occur as in the counting-by-twos pattern, but in a different order.

0 2 4 6 8 0 2 4 6 8 0 2 4 6 8 0 2 4 6 8

Figure 3: The pattern of the last digit of each number in the "counting-by-fours" list.

In the counting-by-sixes column or row $(0, 6, 12, 18, 24, 30, 36, 42, 48, 54, \ldots)$, the pattern of the last digits of each entry is $0, 6, 2, 8, 4, 0, 6, 2, 8, 4, \ldots$. The same last digits occur as in the counting-by-twos pattern, and the counting-by-fours pattern, but in a different order.

0 2 4 6 8 0 2 4 6 8 0 2 4 6 8 0 2 4 6 8

Figure 4: The pattern of the last digit of each number in the "counting-by-sixes" list.

In the counting-by-eights column or row $(0, 8, 16, 24, 32, 40, 48, 56, 64, 72, \ldots)$, the pattern of the last digits of each entry is $0, 8, 6, 4, 2, 0, 8, 6, 4, 2, \ldots$. The same last digits occur as in the counting-by-twos pattern, the counting-by-fours pattern, and the counting-by-sixes pattern, but in a different order.

0 2 4 6 8 0 2 4 6 8 0 2 4 6 8 0 2 4 6 8 0

Figure 5: The pattern of the last digit of each number in the "counting-by-eights" list.

If we summarize the pattern in each of the previous patterns, the same list of last digits appears, albeit in a different order for each of the lists. However, there is an abrupt change

[2] I have started the pattern with the entry 0, because $0 \times 2 = 0$, even though we usually don't put this entry into the multiplication table, perhaps because it's trivial and therefore not worth wasting space in the table. However, in listing patterns, there is an urge for completeness that is apparently stronger than the corresponding one present when compiling tables!

in behaviour in the next list, the counting-by-tens list of numbers: $0, 10, 20, 30, 40, \ldots$. Only the digit 0 appears as the last digit on this list.

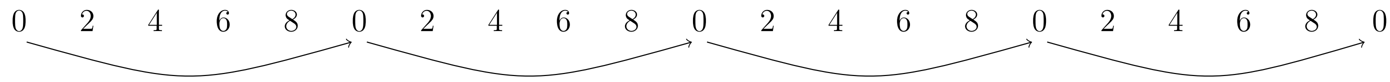

Figure 6: The pattern of the last digit of each number in the "counting-by-tens" list.

Interested students can continue to explore the patterns in the even columns or rows of the multiplication table; that is, in the lists of "counting-by-twelves," "counting-by-fourteens," and so on. Besides providing insight into why the multiples of ten always have 0 as the last digit, one hopes that it will help familiarize students with the final digits of entries in the even columns and rows of the multiplication table.

Continuing, let's examine the patterns in the last digits of the odd columns and rows of the multiplication table. In the counting-by-ones column and row of the multiplication table, each of the digits appears as a last digit, in the natural order: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, \ldots$.

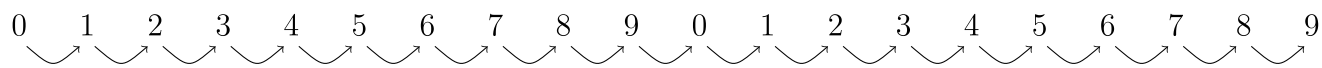

Figure 7: The pattern of the last digit of each number in the "counting-by-ones" list.

In the counting-by-threes list, $0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, \ldots$, each of the digits appears as a last digit, in this order: $0, 3, 6, 9, 2, 5, 8, 1, 4, 7, \ldots$, as illustrated in the following figure:

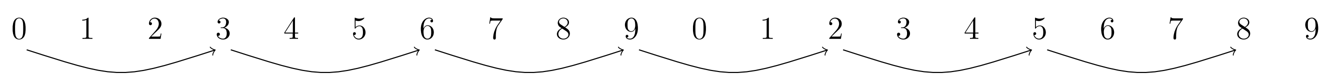

Figure 8: The pattern of the last digit of each number in the "counting-by-threes" list.

In the counting-by-fives list, $0, 5, 10, 15, 20, 25, 30, 35, \ldots$, only the digits 0 and 5 appear, as illustrated in the following figure.

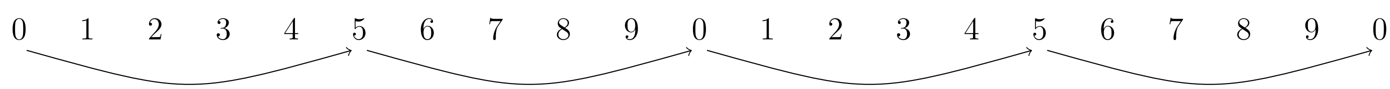

Figure 9: The pattern of the last digit of each number in the "counting-by-fives" list.

The counting-by-sevens list and the counting-by-nines list are similar to the counting-by-threes list, in that all of the digits occur, albeit in a different order. For counting-by-sevens, $0, 7, 14, 21, 28, 35, 42, 49, 56, 63, 70, \ldots$, the order of the last digits is $0, 7, 4, 1, 8, 5, 2, 9, 6, 3, \ldots$. For counting-by-nines, $0, 9, 18, 27, 36, 45, 54, 63, 72, 81, \ldots$, the order of the last digits is $0, 9, 8, 7, 6, 5, 4, 3, 2, 1, \ldots$. The jumps from one number to the next are now so large that diagrams similar to the previous ones may not be very illuminating to students. Perhaps diagrams such as the following "hundreds charts" will be more helpful:

									0
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 10: Counting by sevens; the pattern of last digits is $0, 7, 4, 1, 8, 5, 2, 9, 6, 3, 0, \ldots$.

									0
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 11: Counting by nines; the pattern of last digits is $0, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, \ldots$.

In Figures 10 and 11, the multiples of 7 and 9 (respectively) are highlighted in red. Is it clear from the chart that every digit appears as a last digit in the set of multiples of 7 and 9? After all, there is a red number in each column of the respective hundreds chart.

It might be instructive for students to use a hundreds chart and highlight (or circle) the multiples of 3, or 5, or other numbers.[3] Doing so, and then scanning the chart periodically may be useful in helping students to have instant recall of the multiplication table.

To summarize:

- counting by 2s, 4s, 6s, or 8s: all even numbers (0, 2, 4, 6, and 8) appear as last digits
- counting by 10s: only 0 appears as a last digit
- counting by 5s: only 0 and 5 appear as last digits
- counting by 1s, 3s, 7s, or 9s: all digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) appear as last digits

The Square Numbers

Notice that the square numbers have been highlighted in blue in the multiplication table. A square number results from multiplying a whole number by itself. Visually, you might think of an array of coins (or other items), where the number of coins in each row is equal to the number of rows. Alternatively, you might think of floor tiles in a square room, where the number of tiles in each row is equal to the number of rows, as in the following figures.

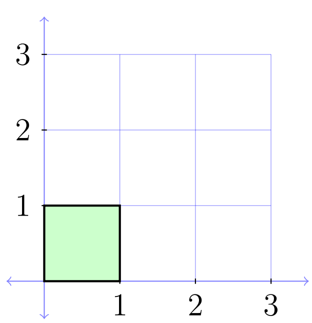

Figure 12: A single floor tile, representing $1 \times 1 = 1$.

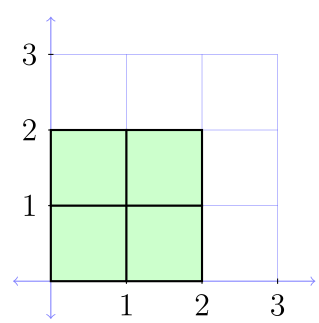

Figure 13: Two rows of floor tiles with two tiles in each row, representing $2 \times 2 = 4$.

[3]Hundreds charts are included at the end of this document should you wish to reproduce them.

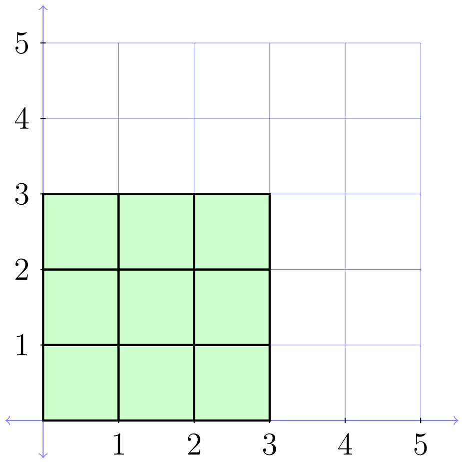

Figure 14: Three rows of floor tiles with three tiles in each row, illustrating $3 \times 3 = 9$.

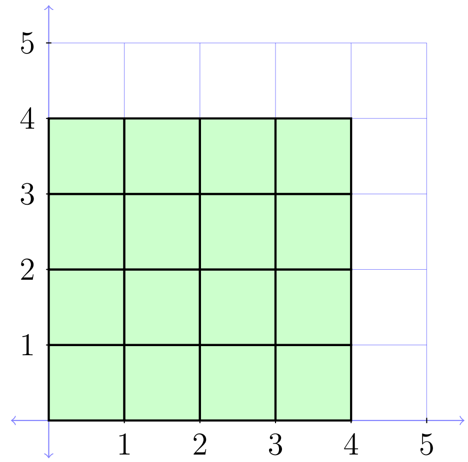

Figure 15: Four rows of floor tiles with four tiles in each row, illustrating $4 \times 4 = 16$.

We could continue with larger and larger arrays of tiles, and by all means students should continue to construct larger square arrays if they wish.

Now that the first few square numbers are illustrated in Figures 12 to 15, imagine that you were actually laying floor tiles (or using blocks, or whatever objects you choose) to gradually build up the four figures, one after the other. That is, what number must be added to each square number to produce the next square number?

It's clear from the multiplication table for the first few square numbers: To go from the first square number to the second square number you add 3 (because $1+3=4$), to continue to the next square number you add 5 (because $4+5=9$), to continue to the next square number you add 7 (because $9+7=16$), and so on. It seems that to go from each square number to the next square number, you add the odd numbers in sequence. In fact, if we include 0 on the list of square numbers (because $0 \times 0 = 0$), then we get all the odd numbers:

$$0 \xrightarrow{+1} 1 \xrightarrow{+3} 4 \xrightarrow{+5} 9 \xrightarrow{+7} 16 \xrightarrow{+9} 25$$

But does this pattern extend indefinitely? If we think that it is valid indefinitely, how could we prove it? Hmmm. While we ponder this, let's go back to the previous four figures and examine the pattern as it manifests in terms of floor tiles arranged in squares.

The first square number, which has just 1 tile, is represented in Figure 12. How many additional tiles do you need to move from Figure 12 to Figure 13? Three tiles are needed; place them in the positions indicated by the crosses in Figure 16 and you will produce the next square number, as illustrated in Figure 17.

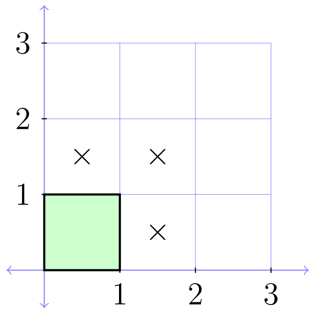

Figure 16: Place 3 additional tiles in the positions indicated by the crosses and . . .

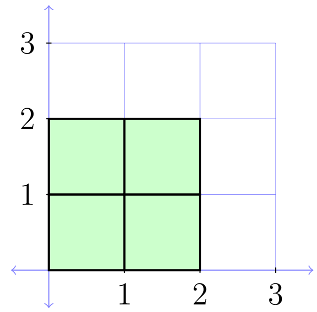

Figure 17: . . . you will obtain the 2×2 square.

Similarly, by placing five tiles in the positions indicated by the crosses in Figure 18 you will obtain the next square number, 3×3, as shown in Figure 19.

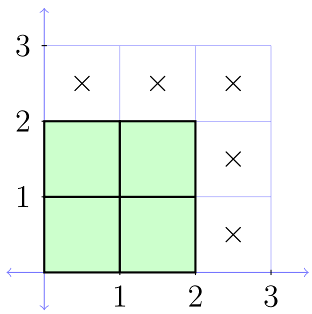

Figure 18: Place 5 additional tiles in the positions indicated by the crosses and . . .

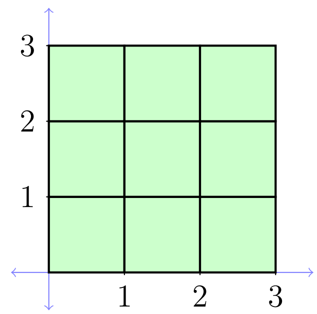

Figure 19: . . . you will obtain the 3×3 square.

To move from each square number to the next larger square number, one must place an L-shaped strip of blocks at the boundary of the previous square of tiles. At each stage, the number of blocks in an L-shaped "boundary strip" is 2 more than the number of tiles in the previous boundary strip, as shown in Figures 20 and 21.

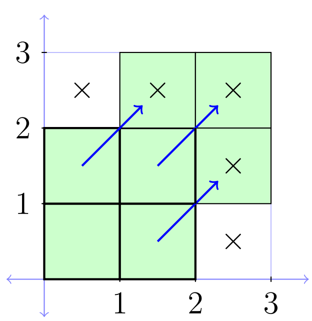

Figure 20: The boundary strip for a 3×3 square contains 5 tiles, which is 2 more than the previous boundary strip.

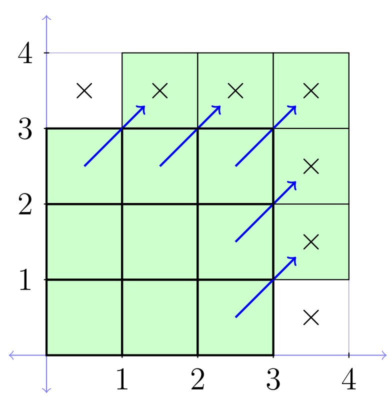

Figure 21: The boundary strip for a 4×4 square contains 7 tiles, which is 2 more than the previous boundary strip.

This might be enough to convince some students that the pattern persists indefinitely, because they might be able to see that in sliding a boundary strip from one square into the boundary of the next larger square (follow the blue diagonal arrows in Figures 20 and 21), there will always be two squares left uncovered, and so two more tiles must be included. This is an argument that each subsequent boundary strip contains two more tiles than the previous one. However, even for such students, one ought to remind them that for the past few centuries the gold standard of mathematical proof is symbolic, and so they should look forward to the day when they are able to construct a symbolic proof.

High-school students should be encouraged to construct such a symbolic proof. Here is one: Consider an $n \times n$ square; that is, the length and width are both made up of n tiles, so that the total number of tiles in the square is n^2. How many tiles are in the square's boundary strip? Well, the number B of tiles in the boundary strip is the difference between the number of tiles in the square and the number of tiles in the smaller square of size $(n-1) \times (n-1)$:

$$\begin{aligned} B &= n^2 - (n-1)^2 \\ &= n^2 - (n-1)(n-1) \\ &= n^2 - (n^2 - 2n + 1) \\ &= n^2 - n^2 + 2n - 1 \\ &= 2n - 1 \end{aligned}$$

So, we've developed a formula for the number of tiles in the boundary strip of an $n \times n$ square. Let's make a table of values to see if they match with the ones we've observed in the multiplication table:

n	$B = 2n - 1$
1	1
2	3
3	5
4	7
5	9
6	11
7	13
8	15
9	17
10	19

The figures in the table match the ones we observe in the multiplication table. Moreover, the difference between consecutive values of B is

$$\begin{aligned} [2(k+1) - 1] - [2k - 1] &= 2k + 2 - 1 - 2k + 1 \\ &= 2 \end{aligned}$$

which shows that the number of tiles in a boundary strip goes up by 2 each time, indefinitely. This proves that the pattern we observed in the multiplication table (that the differences between consecutive values of the square numbers is the sequence of odd numbers) is valid no matter which consecutive square numbers we choose.

Once again, I'd like to emphasize that the symbolic proof just presented can wait for high-school, but elementary-school students ought to be made aware that the proof exists.

That is, an elementary-school student should come away from this exploration with the idea that no matter how many instances are checked, one cannot be sure that a pattern persists indefinitely. Only logical reasoning is convincing, and in mathematics one normally uses a combination of words and symbols to carry out and display logical reasoning, all of which an excited elementary-school student can look forward to one day soon.

Once the pattern of increase of the square numbers is understood, students are empowered to go beyond the table. For instance, moving from 9^2 to 10^2 is an increase of 19, so increasing 10^2 by 21 should take us to the next square number. That is, $11^2 = 10^2+21 = 100+21 = 121$. Similarly, adding another 23 should take us to the next square number, so $12^2 = 11^2 + 23 = 121 + 23 = 144$. These results can be checked by students in whichever ways they choose, which will help them become confident in the results and give them practice in playing with the multiplication table.

The discussion in the previous paragraph suggests that the number of tiles that must be added to an $n \times n$ square to produce an $(n+1) \times (n+1)$ square is $2n+1$. You might like to ask students to illustrate several instances of this fact using tiling diagrams. Can your student see that this must be true in general? (The boundary strip that must be added to an $n \times n$ square to produce the next larger square contains n tiles in the column at the far right, n tiles across the top row, and then 1 additional tile in the new upper right corner; can your student see this? Can your student construct the diagram, or work with manipulatives to see that this is true? If your student is in high-school, can she prove this fact symbolically?)

The "Differences of Squares" Pattern, Part I

Now consider the red numbers in the following multiplication table. Rather than consider the pattern of the numbers within the red diagonals (see the suggestions for further exploration at the end of this document for this), what I'd like to draw your attention to is the relationship between each red number and the closest blue number.

Notice that each of the red numbers is 1 less than the nearest blue number. For example, 3 is 1 less than 4; in other words, $1 \times 3 = 2^2 - 1$. Another example is that 8 is 1 less than 9; that is, $2 \times 4 = 3^2 - 1$. The same pattern is seen in the rest of the table, but is it always true, even for whole numbers that are larger than the ones in the table?

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

If it were always true, then it would be pretty easy to multiply, for example, 19×21. Following the same pattern, $19 \times 21 = 20^2 - 1 = 400 - 1 = 399$. (It's straightforward to calculate 20^2 mentally, because $20 \times 20 = (2 \times 10) \times (2 \times 10) = 2 \times 2 \times 10 \times 10 = 4 \times 100 = 400$.)

A certain kind of student will find this cool!

One might argue that the previous paragraph is pointless. After all, virtually everyone has access to a hand-held calculator nowadays, so there is no need to engage in fancy reasoning; just enter the multiplication problem into a calculator, and that's the end of it. But avoiding the vigorous exercise of reasoning for the torpor of over-reliance on electronic calculators that provide easy answers and no insight doesn't fit into our philosophy. A deep understanding of the patterns in the table, and the development of the skill and confidence to go beyond it, *is* a kind of higher-level thinking.

Back to the pattern in the multiplication table. Is it always true, or does it just happen to be true only for the numbers in the table, which we can see? Let's return to our model of square tiles on a square grid to explore the situation geometrically. Consider $4 \times 4 = 16$ and $3 \times 5 = 15$; how can we make sense of the fact that the latter is 1 less than the former? In terms of tiles, the first fact represents an array of 4 rows with 4 tiles in each row. There are a number of ways to manipulate this array to produce an array of 3 rows with 5 tiles in each row; for instance, you could remove the top row of tiles and slide it over, placing it as a column at the right side of the array. This is illustrated in Figures 22 and 23, where the moved tiles are indicated with a cross.

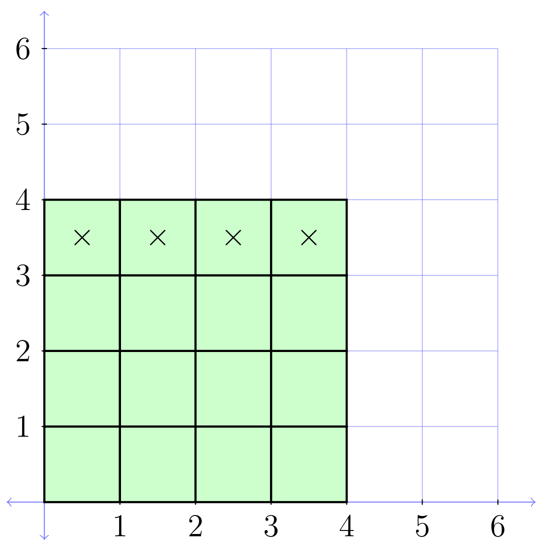

Figure 22: There are 16 tiles arranged in a 4×4 square.

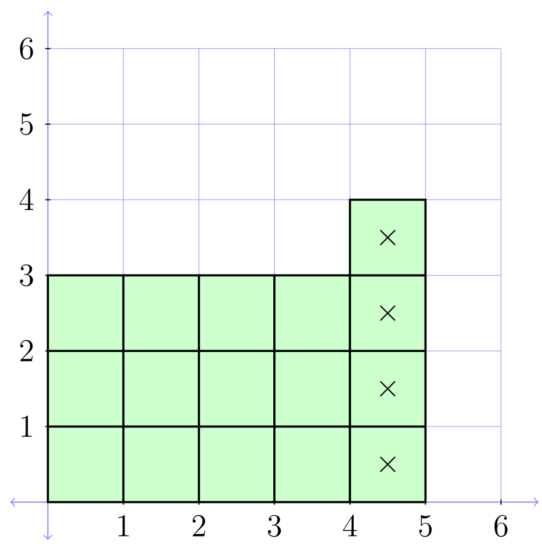

Figure 23: The top row of tiles from the previous figure has been removed and placed as a column at the right side.

Now compare Figures 22 and 23. Notice that Figure 23 is not quite a rectangle, because of the single tile that is "sticking up." However, if you remove this single tile, then we do indeed have a 3×5 rectangle. In other words, removing a single tile from the 4×4 square results in the 3×5 rectangle, which is the same as saying that $3 \times 5 = 4^2 - 1$.

Is it clear that the same will occur no matter what the size of the square is? For example, if we began with a 6×6 square and removed the top row of 6 tiles, there would only be 5 rows left. Then if we slide the 6 tiles over and make them the 7th column, 1 tile will stick out. Therefore, $5 \times 7 = 6^2 - 1$. It is bound to be helpful to students to have them play with manipulatives, constructing squares of various sizes, and then moving a row of tiles to a column so that they can see for themselves that there will always be 1 tile "sticking up."

Playing with tiles or blocks, or drawing diagrams, will be helpful in convincing a student that this pattern will be valid no matter the size of the square, but they should be told that a symbolic proof of this fact is the gold-standard, and will be presented in high-school. Such a proof is as follows:

$$(n-1)(n+1) = n^2 + n - n - 1$$
$$(n-1)(n+1) = n^2 - 1$$

Students should be encouraged to test the formula in the previous line for various values of n to reproduce the patterns seen in the multiplication table. High-school students will encounter this formula as a special case of the "difference of squares" factoring pattern. When elementary students do eventually encounter the difference of squares factoring pattern in high-school they can be reminded of the diagrams and the play with manipulatives that they did here; a potentially scary algebra fact is connected to concrete play with manipulatives and concrete visual images, thereby rendering it familiar and (one hopes) reducing the anxiety that many students feel when confronted by abstractions such as algebra.

The "Differences of Squares" Pattern, Part II

Now consider the red numbers in the following multiplication table. Rather than consider the pattern of the numbers within the red diagonals, once again we'll focus attention on the relationship between each red number and the closest blue number.

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Notice that each of the red numbers is 4 less than the nearest blue number. For example, 5 is 4 less than 9; in other words, $1 \times 5 = 3^2 - 4$. Another example is that 12 is 4 less than 16; that is, $2 \times 6 = 4^2 - 4$. The same pattern is seen in the rest of the table, but is it always true, even for whole numbers that are larger than the ones in the table?

If it were always true, then it would be pretty easy to multiply, for example, 18×22. Following the same pattern, $18 \times 22 = 20^2 - 4 = 400 - 4 = 396$. Similarly, $98 \times 102 = 100^2 - 4 = 10\,000 - 4 = 9\,996$. As usual, students can be encouraged to check these results in various ways, either by paper-and-pencil calculations, with a hand-calculator, or both.

Is it strange that one must subtract 4 in this case, whereas in the previous situation one subtracted 1? It certainly calls out for further understanding. And what about the geometric perspective? OK, let's construct a diagram of tiles, as before, to see what insight this brings. (Of course, you can also use manipulatives such as coins or blocks to achieve the same ends.) To be specific, let's start with an array of 4 rows each containing 4 tiles, so that we have a 4×4 square. Then let's remove the top two rows of tiles, moving them to the right and placing them as columns, as illustrated in Figures 24 and 25. The tiles that are moved are marked with a cross.

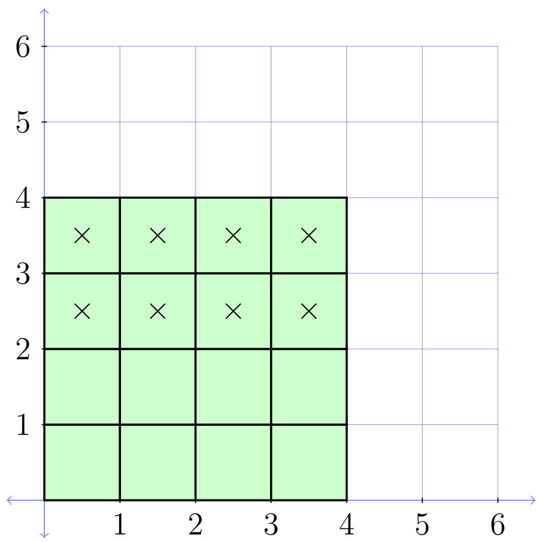

Figure 24: There are 16 tiles arranged in a 4×4 square.

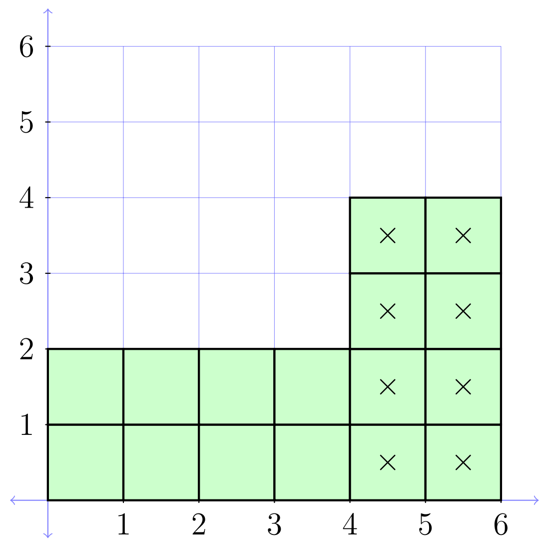

Figure 25: The top 2 rows of tiles from the previous figure have been removed and placed as columns at the right side.

Aha! There it is: Instead of a single tile "sticking up" in Figure 25, there is a 2×2 square of tiles "sticking up" on top of the 2×6 rectangle. Thus, starting from the 4×4 square, one must take away a 2×2 square so that a 2×6 rectangle remains: $4^2 - 2^2 = 2 \times 6$.

Students can now be encouraged to play with starting squares of other sizes, taking away two rows of tiles and placing them as columns, so that they can see that there is always a 2×2 square of tiles "sticking up" above the resulting rectangle. A symbolic proof that this is always true, no matter what the size of the starting square, which is suitable for high-school students, is:

$$(n-2)(n+2) = n^2 + 2n - 2n - 4$$
$$(n-2)(n+2) = n^2 - 4$$
$$(n-2)(n+2) = n^2 - 2^2$$

Once again, the formula in the previous line is a special case of the difference of squares factoring pattern that students will study in high school algebra.

The "Differences of Squares" Pattern, Part III

OK, now a good mathematical habit of mind is that of *generalization*. That is, now that we've seen a pattern that involves stripping one row of tiles from a square and placing it as a column, then stripping two rows of tiles and placing them both as columns, a natural question is, "Are there similar patterns if one strips off other numbers of rows and places them as columns?"[4] If so, then all of the patterns for each individual numbers of rows can be thought of collectively as forming parts of one grand pattern. The answer to the question is "Yes." and the collective pattern is called the difference of squares factoring pattern.

That is to say, if you begin with an array of n rows of tiles with n tiles in each row, so that they form an $n \times n$ square, and you strip off k rows of tiles, placing them as columns to the right of the figure, then you end up with a rectangle of dimensions $(n-k) \times (n+k)$, with an additional $k \times k$ square "sticking up" at the top of the figure. This can be proven symbolically, as we've done before, as follows:

$$(n-k)(n+k) = n^2 + nk - nk - k^2$$
$$(n-k)(n+k) = n^2 - k^2$$

The formula in the previous line is the difference of squares factoring pattern. As usual, proofs such as this one can be saved for high school. Elementary students can be encouraged to explore the situation, however, by starting with squares of various sizes, stripping off various numbers of rows of tiles, examining the resulting rectangle (together with the additional small square of tiles "sticking up"), and finally correlating their work with the appropriate locations in the multiplication table. And students can also play with the formula, if they are at the appropriate developmental level to be able to do so, by substituting various values for n and k, and comparing their results to those obtained by manipulating arrays of coins or blocks, or drawing tiling diagrams similar to the ones illustrated in this document.

The Distributive Property

The distributive property allows one to relate an entry in the multiplication table to a neighbouring one in the same row or column. Once part of the table is learned, one can use the distributive property to calculate neighbouring entries of the table. For example, suppose that the square numbers are already memorized, so that a student already knows instantly that $7 \times 7 = 49$. Then it is relatively easy to calculate 8×7, because (by the distributive property)

$$\begin{aligned} 8 \times 7 &= (7+1) \times 7 \\ &= 7 \times 7 + 1 \times 7 \\ &= 49 + 7 \\ &= 56 \end{aligned}$$

In other words, eight sevens is equal to seven sevens plus one additional seven. Thus, if you know that $7 \times 7 = 49$, you can determine the value of 8×7 by simply adding 7 to 49, without having to calculate 8×7 from scratch.

A Practical Approach to Learning the Multiplication Table

If I were embarking on a program to help a student memorize the multiplication table, I would begin by playing with the patterns illustrated in this document, and then encouraging the student to discover other patterns. Regular, consistent play with the table will make it

[4] If a student thinks of such a question herself, that's great! However, that's not the most important thing. What is important is to ask the question if a student does not think of it himself, and to ask it a sufficient number of times that the urge to generalize (along with the urge to ask questions!) becomes a habit.

meaningful, and therefore it will be easier to develop instant recall of the entries, which will be very useful. But beyond this, it will help prepare the student for future developments.

If some of the play includes counting, then I would begin with counting by twos, then counting by tens, and then counting by fives. Next I would play with counting by twos, fours, and eights, and the relations among them. Students should notice that every second number in the counting-by-twos list appears in the counting-by-fours list, and similarly, every second number in the counting-by-fours list appears in the counting-by-eights list. Alternatively, every fourth number in the counting-by-twos list appears in the counting-by-eights list.

Next I would play with counting by threes, sixes, and nines, and the relations among them. Students should notice that the every second number in the counting-by-threes list appears in the counting-by-sixes list, and every third number in the counting-by-threes list appears in the counting-by-nines list.

Next I would play with the counting-by-sevens list.

Once the various "counting-by" lists are mastered, one can always "fill in" entries that are not well-known yet by applying the distributive property to move from well-known entries to adjacent entries that are not well-known.

Concluding Remarks

That the multiplication table should be memorized is a fundamental assumption of this document, based on my experience in teaching mathematics for many years. If certain basic skills are known so deeply that they are "automatic" then learning advanced material is possible and smoother. On the other hand, a lack of mastery of basic skills makes learning advanced material a stressful, anxious, painful nightmare, particularly with our current system of higher education.

Learning the multiplication table by rote memorization ought to be the absolute last resort, because by learning the table in this way, students will know only these few facts, and nothing more. I highly recommend *playing* with the material, as illustrated in this document (or in whatever other way that you dream up — in fact, it's probably better for you and your students if you do dream up your own ways), because your students will learn so much more than just the multiplication table. For instance, your students will get good examples of, and practice in

- asking questions
- the important mathematical habit of generalizing
- making connections between (future) abstract, symbolic perspectives, with concrete visual or manipulative perspectives, a habit that will serve them well in all of their studies of mathematics and science

By helping your students play with their mathematics, you will be planting seeds so that good thinking habits will grow. By playing with the multiplication table, students will understand it deeply, see many connections, and will remember the table automatically, without any further effort. The memorization will be a happy *byproduct* of the play.

"Knowledge keeps no better than fish," said Alfred North Whitehead, and so it's better to promote understanding, because that which is understood will be remembered more effectively, more usefully, and with more pleasure, than that which one is force-fed with no understanding. It's better to play, so that one develops good habits of thought, and so that one becomes friends with the subject matter.

And if after much play a student still does not have instant recall of the multiplication table, then by all means use rote memorization; but please use this as an absolute last resort. And even then, there are ways to make this task fun, by making a game out of it. Coercion just breeds hatred, so if a child is really resistant, maybe we should not be forcing the child to learn mathematics just at the moment, but rather we should be facilitating the child doing whatever causes his heart to sing.

Remember: If you memorize a math fact, then all you know is that one fact. Memorizing the multiplication table leaves you with just that and nothing more. However, if you understand the patterns that underlie the multiplication table, then you are empowered to go far beyond the confines of the table. And you are well-prepared to successfully tackle advanced work, because the fundamental ideas of mathematics recur over and over again; in different guises and contexts perhaps, but the ideas are still there.

There are other patterns in the multiplication table, and if you and your students are interested, continue to play with the table. (See the exercises listed below for ideas.) Encourage students to come up with diagrams that explain the patterns. In lieu of diagrams, construct models that can be manipulated (using rows of coins, or blocks, or whatever material is easily available). Then see if you can construct a formula that represents the pattern symbolically; proving that the formula is an identity will prove that the pattern is valid for all whole numbers, not just the ones you studied in the table. It will also help to connect the pattern to high-school algebra.

Note that I am not suggesting that elementary school children work with formulas! The formulas are for your own purposes as a teacher, as they will help you make connections between elementary mathematics and high-school mathematics. If you have the occasion to teach high-school algebra, you will have additional tools in your kit — you will be able to encourage high-school students to make abstract algebraic identities concrete using the multiplication table or with diagrams or models. Working with concrete examples or models is one of the major ways to cope with abstraction, and encouraging students to make their own concrete examples and models is bound to be helpful.

Further Explorations of Patterns in the Multiplication Table

If you and your students are interested in further explorations of patterns in the multiplication table, here are some ideas to get you started.

1. We saw earlier that in moving from one square number to the next one (i.e., moving along the main diagonal of the multiplication table), the squares increase by the sequence of odd numbers. What if, instead of moving from each square number to the next one, you jumped so that you skipped alternate square numbers. Is there a pattern in the difference between the square numbers if you skip alternate ones? What if you skip two consecutive square numbers, so that you only land on every third number? What if you only land on every fourth number?

2. We saw earlier that in moving from one square number to the next one (i.e., moving along the main diagonal of the multiplication table), the squares increase by the sequence of odd numbers. What if, instead of moving along the main diagonal (the square number, indicated in blue), we moved along the neighbouring parallel diagonal starting with the number 2 (i.e., $2, 6, 12, \ldots$); what is the pattern for this diagonal? What if we moved along a parallel diagonal starting with the number 3 (i.e., $3, 8, 15, \ldots$); what is the pattern for this diagonal? What happens for other parallel diagonals?

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41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
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91	92	93	94	95	96	97	98	99	100

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91	92	93	94	95	96	97	98	99	100

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