The Resistor Cube Problem

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It's a relaxing Saturday afternoon at your local amateur radio club and you're busy discussing the finer points of antenna theory with some friends. The next thing you know a fellow amateur interrupts you by throwing you a small neatly soldered resistor cube. This person then challenges you to find the resistance between any two diagonally opposite corners without using an ohmmeter.

Problem: The resistor cube consists of 12 resistors, each having a value of exactly 1 ohm. What is the total resistance between the two diagonally opposite corners labeled A and H on the diagram below?
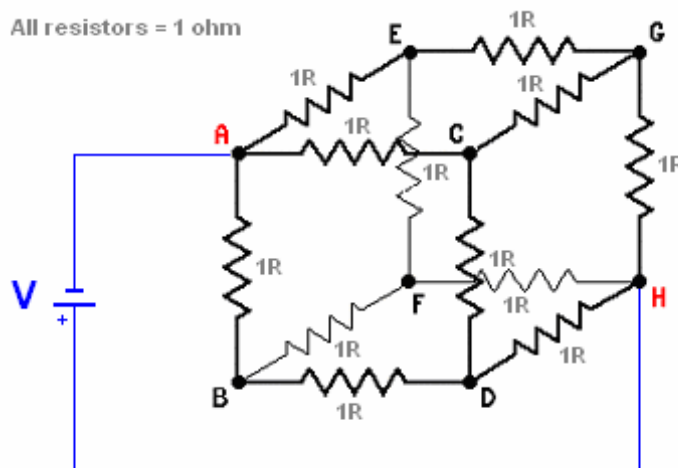

Figure 1

Solution: There are many different approaches to solving this problem and I would encourage you to have a go at solving it for yourself before looking at the solutions I have presented.

On the pages that follow I show two very different methods for solving this particular problem. The first method, which is a more generic method, uses loop equations and simultaneous equations to solve the problem. The second method is a very elegant solution and shows what can be achieved by using a totally different approach. It makes use of simple logical reasoning with some ohms law thrown in.

Solution 1 – Using Loop Equations

The basic steps for solving this problem using loop equations are:

1. Redraw the 3 dimensional resistive cube network in 2 dimensions.
2. Draw all the loop currents in a clockwise direction and identify them.
3. Write the equations for the voltage drops around each loop in turn.
4. Solve the equations to find the unknown currents using the method of simultaneous equations.

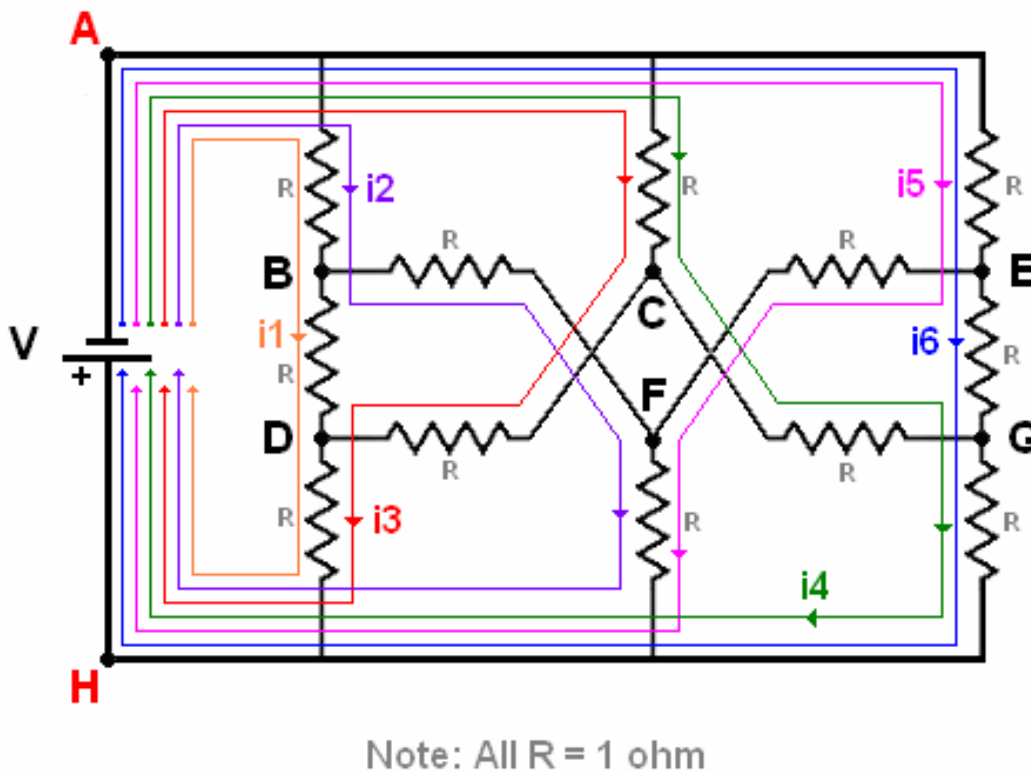**Step 1 & 2:** Draw the circuit in 2d and identify the loop currents.

Note: All R = 1 ohm

Figure 2

Step 3: Write the voltage drop equations for each of the loops.

Loop i_1 (orange):

$V = (I_1 + I_2)R + I_1R + (I_1 + I_3)R$ => Multiply out the brackets.

$\quad = I_1R + I_2R + I_1R + I_1R + I_3R$ => Because R = 1 we can simplify to:

$\quad = I_1 + I_2 + I_1 + I_1 + I_3$ => Collect like terms.

$\quad = 3I_1 + I_2 + I_3$

Loop i2 (purple):

$V = (I_1 + I_2)R + I_2R + (I_2 + I_5)R$ => Multiply out the brackets.

$= I_1R + I_2R + I_2R + I_2R + I_5R$ => Because R = 1 we can simplify to:

$= I_1 + I_2 + I_2 + I_2 + I_5$ => Collect like terms.

$= I_1 + 3I_2 + I_5$

Loop i3 (red):

$V = (I_3 + I_4)R + I_3R + (I_1 + I_3)R$ => Multiply out the brackets.

$= I_3R + I_4R + I_3R + I_1R + I_3R$ => Because R = 1 we can simplify to:

$= I_3 + I_4 + I_3 + I_1 + I_3$ => Collect like terms.

$= I_1 + 3I_3 + I_4$

Loop i4 (green):

$V = (I_3 + I_4)R + I_4R + (I_4 + I_6)R$ => Multiply out the brackets.

$= I_3R + I_4R + I_4R + I_4R + I_6R$ => Because R = 1 we can simplify to:

$= I_3 + I_4 + I_4 + I_4 + I_6$ => Collect like terms.

$= I_3 + 3I_4 + I_6$

Loop i5 (pink):

$V = (I_5 + I_6)R + I_5R + (I_5 + I_2)R$ => Multiply out the brackets.

$= I_5R + I_6R + I_5R + I_5R + I_2R$ => Because R = 1 we can simplify to:

$= I_5 + I_6 + I_5 + I_5 + I_2$ => Collect like terms.

$= I_2 + 3I_5 + I_6$

Loop i6 (blue):

$V = (I_5 + I_6)R + I_6R + (I_4 + I_6)R$ => Multiply out the brackets.

$= I_5R + I_6R + I_6R + I_4R + I_6R$ => Because R = 1 we can simplify to:

$= I_5 + I_6 + I_6 + I_6 + I_4$ => Collect like terms.

$= I_4 + I_5 + 3I_6$

We now have six equations with six unknowns, I_1 to I_6, so we can solve this problem using the method of simultaneous equations. We begin by assigning some value of voltage to V and then writing out the six equations.

If we let the voltage V = 1 Volt then the six equations are:

$$1 = 3I_1 + I_2 + I_3$$
$$1 = I_1 + 3I_2 + I_5$$
$$1 = I_1 + 3I_3 + I_4$$
$$1 = I_3 + 3I_4 + I_6$$
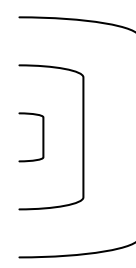$$1 = I_2 + 3I_5 + I_6$$
$$1 = I_4 + I_5 + 3I_6$$

Now solving six simultaneous equations is not an easy task, but luckily for us this particular problem allows us to reduce the number of equations involved. We can do this because of the symmetry of the resistor cube network and the fact that all the resistors are identical in value. If you look at the two-dimensional diagram in figure 2 you should be able to see that the following currents must be equal.

$$I_1 = I_6$$
$$I_2 = I_5$$
$$I_3 = I_4$$

Substituting the currents I_4, I_5 and I_6 with their equivalents I_3, I_2 and I_1 in the above six equations yields:

$$1 = 3I_1 + I_2 + I_3$$
$$1 = I_1 + 3I_2 + I_2$$
$$1 = I_1 + 3I_3 + I_3$$
$$1 = I_3 + 3I_3 + I_1$$
$$1 = I_2 + 3I_2 + I_1$$
$$1 = I_3 + I_2 + 3I_1$$

Notice that due to the symmetry we have pairs of identical equations, this is good news as it now means that we only have 3 equations with 3 unknowns to solve.

So, the 3 equations are:

$$1 = 3I_1 + I_2 + I_3$$
$$1 = I_1 + 3I_2 + I_2$$
$$1 = I_1 + 3I_3 + I_3$$

If you look carefully at the 3 above equations you will notice that they can be further simplified by collecting like terms.

$1 = 3I_1 + I_2 + I_3$
$1 = I_1 + 4I_2$
$1 = I_1 + 4I_3$

We can now begin to solve the currents I_1, I_2 and I_3 using the method of simultaneous equations.

$1 = 3I_1 + I_2 + I_3$ (Equation E1)
$1 = I_1 + 4I_2$ (Equation E2)
$1 = I_1 + 4I_3$ (Equation E3)

Multiply Equation E1 by 4.
(E1) x 4   => $4 = 12I_1 + 4I_2 + I_3$ (Equation E4)

Subtract equation E2 from equation E4 to cancel the I2 terms.
(E4) - (E2)   => $3 = 11I_1 + 4I_3$ (Equation E5)

Subtract equation E3 from equation E5 to cancel the I3 terms.   Then solve for I1.
(E5) - (E3)   => $2 = 10I_1$

$2 = 10I1$
$\therefore I1 = 2 / 10$
$\therefore I1 = 1/5A$

Solve for I2 by substituting I1 into equation (E2)
$1 = 1/5 + 4I_2$
$\therefore I_2 = (1 - 1/5) / 4$
$\therefore I_2 = \underline{1/5A}$

Solve for I3 by substituting I1 into equation (E3)
$1 = 1/5 + 4I_3$
$\therefore I_3 = (1 - 1/5) / 4$
$\therefore I_3 = \underline{1/5A}$

We have now shown that the currents I_1, I_2 and I_3 are equal and have a value of 1/5A. But what about the other three currents I_4, I_5 and I_6? Well remember earlier we said that because of the symmetry of the cube, the currents $I_1 = I_6$, $I_2 = I_5$ and $I_3 = I_4$. Therefore we can conclude that all of the currents I_1 through to I_6 are the same and equal to 1/5A.

Therefore the total current flowing into the circuit is:

$$I_{TOTAL} = I_1 + I_2 + I_3 + I_4 + I_5 + I_6$$
$$= 1/5 + 1/5 + 1/5 + 1/5 + 1/5 + 1/5$$
$$= \underline{6/5A}$$

We now use Ohm's law to determine the total resistance of the cube resistor network.

$$R_T = E / I_T$$
$$= 1V / (6/5A)$$
$$= \underline{5/6\ \Omega}$$

The total resistance between the two diagonally opposite corners labelled A and H on the diagram in figure 1 is 5/6 ohms.

Solution 2 – Using Logical Reasoning

The basic steps needed to solve this problem are:

1. Assume a current of 1 ampere is flowing into the circuit.

2. Determine the current flowing in each branch of the network.

3. Determine the voltage dropped across each resistor in the network and so determine the applied voltage V.

4. Use ohms law to calculate the total resistance of the network.

Step 1:

The first step towards solving the total resistance of the cube is to assume that a current of 1A is flowing into the resistor cube network. The diagram below shows 1A entering the network at junction A and 1A leaving it at H.
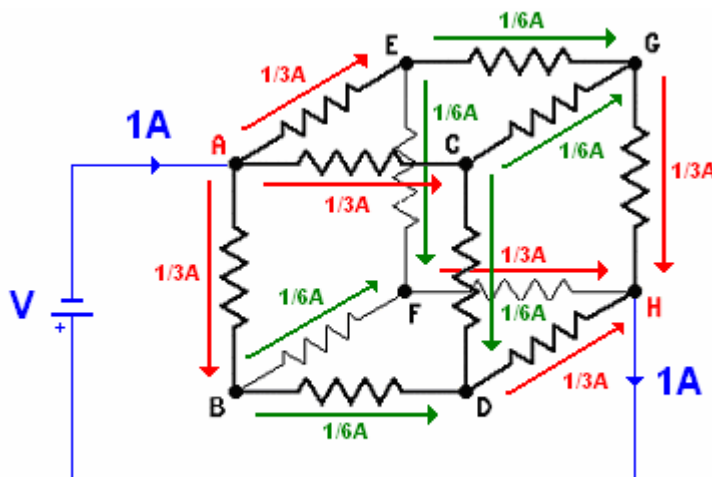

Figure 3

Step 2:

Now because of the symmetry of the resistor cube network, the 1A current flowing into junction A must divide evenly between the 3 branches AE, AC and AB so each branch will have 1/3A flowing in it.

Now the currents in AE, AC and AB each divide evenly into two further branches at junctions E, C and B. This means we have 1/6A flowing through the resistors in branches BD, BF, EG, EF and CD, CG, shown in green above.

The six 1/6A branches then recombine in pairs at junctions D, F and G so we have 1/3A flowing in branches DH, FH and GH.

These three branches then recombine at junction H and 1A flows out of the resistor cube network.

Step 3:

We have now determined the value of the current flowing in each branch of the cube network. We also know that each resistor has a value of 1 ohm, therefore we can calculate the voltage dropped across each resistor in the network. All we then have to do is sum the voltage drops of any single path between A and H.

Using the path A, B, D, H we see that:

$$V = (1/3A)R + (1/6A)R + (1/3A)R$$
$$= (1/3A \times 1\Omega) + (1/6A \times 1\Omega) + (1/3A \times 1\Omega)$$
$$= 1/3 + 1/6 + 1/3$$
$$= \underline{5/6V}$$

The applied voltage V is equal to 5/6 volts.

Step 4:

We now know the applied voltage V along with the value of the current I flowing into the circuit, so we can use ohms law to calculate the total resistance of the R_T.

$$R_T = V / I$$
$$= (5/6V) / 1A$$
$$= \underline{5/6\Omega}$$

The total resistance between the two diagonally opposite corners labelled A and H on the diagram in figure 1 is 5/6 ohms.